The Cereal Box Problem Revisited

Jesse L. M. Wilkins
Virginia Polytechnic Institute and State University

The Cereal Box problem is fascinating for students of all levels. Rich in mathematical content, this problem offers students the opportunity to collect data, make conjectures, and derive mathematical models. Using Monte Carlo methods, the Cereal Box problem is investigated in this paper, using both an experimental and theoretical framework. This investigation extends previous considerations of the Cereal Box problem. Using empirical data, students can discover patterns and relationships that help them understand the origin of the theoretical solution to the problem. Building on experimental findings, a theoretical model is derived, showing that the expected solution of the Cereal Box problem is formed from the sum of successive geometric series.

In order to understand the theoretical underpinnings of probability, students need opportunities to develop their probabilistic intuition through empirical investigations that can motivate and help construct sound probabilistic understanding (Konald, 1991, 1994; National Council of Teachers of Mathematics [NCTM], 1989). In other words, students should be actively involved with hands-on experiments of uncertainty—at the basics, flipping pennies and rolling dice. With increased emphasis from the NCTM to involve students in the creation of both experimental and theoretical models to analyze and interpret real-world situations of uncertainty also comes an increased need for problems and activities allowing for such investigation and modeling.

The Cereal Box problem (Travers, 1981) can be investigated by high-school students (Litwiller & Duncan, 1992; Maull & Berry, 1997; Travers & Gray, 1981; Travers, Stout, Swift, & Sextro, 1985), middle-school students (Lappan & Winters, 1980; Riddiough & McColl, 1996), and even students as young as fifth grade (Richbart & Richbart, 1992; Zawojewski, 1991):

> Munchy Crunch Cereal offers a pen of one of six different colors in a box. Assuming equal chances of getting any of the six colors of pen with one purchase, how many boxes of Munchy Crunch would one expect to have to buy to obtain the complete set of six pens? (Travers, 1981, p. 210)

Most students at one time or another have tried to obtain a set of free prizes offered in their favorite cereal. The problem offers a real-world situation in which students can collect their own data and derive both interesting experimental and theoretical solutions. Teachers can adapt this problem for all levels of students. Younger students can solve the problem intuitively and informally through experimental simulations. These experiments can be extended further to motivate older students to delve deeper into the problem and look for patterns that may help derive theoretical models.

The Cereal Box problem is often used to introduce the concept of expected value and to show the power of the Monte Carlo method (e.g., Travers et al., 1985). However, many past activities using the Cereal Box problem help students estimate the expected value and leave the student with the theoretical answer (14.7 boxes) for comparison (Riddiough and McColl, 1996; Travers, 1981; Travers et. al., 1985; Zawojewski, 1991). Lappan and Winter (1980) provided a derivation of the theoretical value of 14.7 boxes. However, the derivation was built on the undiscovered or unproven fact that, if an event has a probability p of occurring, on average it will take $1/p$ trials for it to occur. Mosteller (1965) offered a derivation of this fact. The remainder of this paper describes a series of activities that can be used to guide students through an experimental and inquiry-based investigation of why 14.7 boxes is the expected solution to the problem. Finally, by building on discoveries from these experiments and incorporating ideas associated with the geometric distribution and sums of geometric series, derivations similar to those of Mostellar and Lappan and Winter are constructed to create a complete derivation.

An Experimental Investigation

Using a Die

After a little thought, students often decide that an estimate of the number of boxes needed to obtain all six prizes (colored pens) can be obtained by actually buying cereal. They could go to the store, buy many boxes of cereal, and keep track of the prizes obtained until they get all six. This shopping spree would then be repeated several times in hopes of finding some sort of pattern. However, this type of experiment can get very expensive. At this point students might suggest a random device, such as a die, in place of the expensive shopping spree. Using such a device to produce random outcomes in order to investigate a given problem is known as a Monte Carlo simulation and can range in form from rolling dice and flipping coins to elaborate computer programs capable of doing thousands of trials.

A Monte Carlo simulation for the Cereal Box problem might proceed as follows (for further explanation of the Monte Carlo methods, see Shulte & Smart, 1981; Sobol′, 1994; Travers, 1981; Travers et al., 1985):

1. Identify the model. A six-sided die makes a good model for the problem. The six faces or numbers on the die can represent the six different-colored prizes. A roll of the die can represent the purchase of one box of cereal.

2. Define a trial. A trial ("shopping spree") consists of students rolling the die until they obtain all six different faces or numbers on the die ("prizes").

3. Record the statistic of interest. For each trial, record in a table the number of rolls required to get all six numbers (prizes). In Table 1, subcounts are made of each outcome. These are summed to obtain the total number of rolls.

4. Repeat trials. To obtain a relatively accurate estimate, do at least 100 trials. In a class of 30 students, 100 trials can be quickly obtained.

5. Find an average. Average the number of purchases for all trials (shopping sprees) to estimate the expected number of box purchases necessary to get all six prizes.

Table 1 summarizes the results of five trials using a six-sided die, where *N* represents the total number of rolls per trial to obtain all six prizes. The overall average of 15.6 boxes represents an estimate of the expected number of boxes to buy. A discussion of this estimate might include the reality of a store owner allowing a customer to buy 0.6 boxes of cereal. Thus, students might conclude from this experiment that, on average, it would be necessary to buy 16 boxes of cereal.

Using a Computer

Computers enable students to do many thousands of trials very quickly to obtain an even better estimate of the number of boxes to buy. In the Cereal Box problem, the model for purchasing cereal and obtaining all six prizes could be represented by having the computer randomly choose from the set of integers 1 to 6. The remaining steps of the Monte Carlo simulation can also be programmed by the student for the computer to process. Figure 1 presents a histogram from 15,000 trials of such a program. The histogram represents the distribution of the number of boxes purchased. Based on this simulation, the mean number of boxes purchased was 14.68. In other words, one would expect to have to buy 14.68 boxes of cereal to get all six prizes.

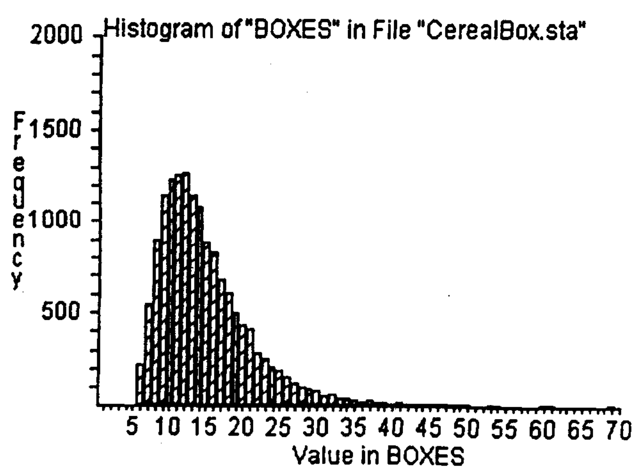
The Cereal Box problem simulation can be programmed using many computer languages (e. g., BASIC, C+). Resampling Stats (Simon, Weidenfeld, Bruce, & Puig, 1995) was used for the simulations in this

Table 1
Results From Five Trials Using a Six-Sided Die to Represent the Different Colored Prizes.

<table>
<tr><th rowspan="2">Shopping Spree (Trial)</th><th colspan="6">Prizes</th><th rowspan="2">N</th></tr>
<tr><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th></tr>
<tr><td>1</td><td>|||</td><td>|||.</td><td>|</td><td>|||</td><td>||||</td><td>|||||</td><td>19</td></tr>
<tr><td>2</td><td>|||</td><td>||||| ||</td><td>|</td><td>||||</td><td>|||</td><td>|||</td><td>21</td></tr>
<tr><td>3</td><td>||||| |</td><td>|</td><td>|</td><td>|</td><td>|</td><td>||</td><td>12</td></tr>
<tr><td>4</td><td>||</td><td>|</td><td>|||</td><td>||</td><td>|</td><td>|</td><td>10</td></tr>
<tr><td>5</td><td>|</td><td>||||</td><td>|</td><td>||</td><td>||||| |</td><td>||</td><td>16</td></tr>
</table>

Note. The average number of boxes of cereal purchased would be (19+21+12+10+16)/5 = 15.6 boxes.

Figure 1. *Distribution of 15,000 estimates of the expected value of the Cereal Box problem using Monte Carlo computer simulation.*



Note: Overall mean is 14.68.

article because its structure is well suited for Monte Carlo methods. The programs referred to in this article can be found by contacting the author or at http://www.chre.vt.edu/F-S/wilkins/CerealBox/CerealBox.htm.

Past investigations of the Cereal Box problem often leave the problem at this point, offering the theoretical answer of 14.7 boxes as a point of comparison (e.g., Travers, et. al., 1985; Zawojewski, 1991). As can be seen, the previous estimates are quite good; however, this paper offers an extension of the above simulation that can help lead middle- and high-school students to discover the patterns making up the formula for calculating the expected number of boxes, that is,

$$\frac{6}{6}+\frac{6}{5}+\frac{6}{4}+\frac{6}{3}+\frac{6}{2}+\frac{6}{1}=14.7.$$

For more advanced students, this investigation can eventually lead to a proof that the previous calculation gives the expected value for obtaining all six prizes and can also lead to a general formula for any number of prizes.

An Extension of the Experimental Investigation

Consider a similar question associated with the Cereal Box problem: What is the expected number of purchases required to obtain a prize different from the first prize? This question can be extended further to the case of obtaining a different prize after getting the first two prizes and so on until the case of getting the last prize after successfully obtaining the first five prizes.

Using a Die

Using a die to represent the six prizes, suppose that the first roll of the die is a 5. On average, how many subsequent rolls would be expected before rolling a number different from 5? To continue, suppose the first two rolls are 5 and 2. On average, how many subsequent rolls would be expected before obtaining something other than 5 or 2? Finally, an estimate of the number of rolls required to get all six individual prizes in succession is obtained.

Once again, Monte Carlo methods can be used to answer these questions. For this experiment, a trial now consists of rolling a die until obtaining a prize different from those already obtained. This experiment can be carried out individually for each number of previous prizes obtained. However, this experiment could also be carried out in conjunction with the original experiment, with each "trial" nested within the larger trial of finding the total number of rolls necessary for obtaining all six prizes. Table 2 summarizes the results of five such nested trials. The column pairs represent a trial of getting all six prizes. Each row gives the results of trying to obtain a different prize after first obtaining a given number of prizes.

For example, in Trial 1, the first prize was a 2, and it took one roll to obtain it. The second prize (different from 2) was a 4, and it was obtained in two rolls. The third prize (different from 2 and 4) was a 1, and it was obtained in one roll. The fourth, fifth, and sixth new prizes were 3, 6, and 5 in that order, taking four, one, and two rolls, respectively. In total, it took 11 rolls to obtain all six prizes. The Mean Number of Rolls column in Table 2 gives estimates of the expected values for each of the nested experiments, as well as an overall estimate of the number of rolls to get all six prizes. Based on this experiment, on average it takes one roll to get the first prize, 1.2 additional rolls to get the second prize, 1.4 additional rolls to get the third prize, etc. Once again by having each student do several trials, the class's estimate becomes more accurate, and noticeable patterns begin to form. For

Table 2

Results From Five (Nested) Trials Estimating the Number of Rolls Expected for Each Individual Prize in Succession.

<table>
<tr><th rowspan="3"># of Prize</th><th colspan="10">Trials</th><th rowspan="3">Total Number of Rolls</th><th rowspan="3">Mean Number of Rolls</th></tr>
<tr><th colspan="2">Trial 1</th><th colspan="2">Trial 2</th><th colspan="2">Trial 3</th><th colspan="2">Trial 4</th><th colspan="2">Trial 5</th></tr>
<tr><th>Prize</th><th>Tally of Rolls</th><th>Prize</th><th>Tally of Rolls</th><th>Prize</th><th>Tally of Rolls</th><th>Prize</th><th>Tally of Rolls</th><th>Prize</th><th>Tally of Rolls</th></tr>
<tr><td>1</td><td>2</td><td>|</td><td>1</td><td>|</td><td>4</td><td>|</td><td>3</td><td>|</td><td>2</td><td>|</td><td>5</td><td>1</td></tr>
<tr><td>2</td><td>4</td><td>||</td><td>3</td><td>|</td><td>6</td><td>|</td><td>1</td><td>|</td><td>6</td><td>|</td><td>6</td><td>1.2</td></tr>
<tr><td>3</td><td>1</td><td>|</td><td>2</td><td>||</td><td>2</td><td>|</td><td>4</td><td>|</td><td>4</td><td>||</td><td>7</td><td>1.4</td></tr>
<tr><td>4</td><td>3</td><td>||||</td><td>5</td><td>||</td><td>5</td><td>|</td><td>6</td><td>||</td><td>3</td><td>||</td><td>11</td><td>2.2</td></tr>
<tr><td>5</td><td>6</td><td>|</td><td>4</td><td>||||| ||</td><td>3</td><td>|</td><td>2</td><td>||||| ||</td><td>1</td><td>|</td><td>17</td><td>3.4</td></tr>
<tr><td>6</td><td>5</td><td>||</td><td>6</td><td>|||</td><td>1</td><td>||||| |</td><td>5</td><td>||||| |</td><td>5</td><td>||||| ||</td><td>24</td><td>4.8</td></tr>
<tr><td>Total</td><td></td><td>11</td><td></td><td>16</td><td></td><td>11</td><td></td><td>18</td><td></td><td>23</td><td>70</td><td>14.0</td></tr>
</table>

example, students may notice that as one obtains more prizes it becomes harder to get the later prizes.

At this point, students might be encouraged to think about the probability of attaining a new prize after first getting a certain number of prizes. For example, have them figure out the probability of rolling a different number after first rolling a 5 and a 2. To facilitate later discoveries, have them put their calculations in a table like Table 3, and also have them calculate the reciprocal of the probabilities and discuss any patterns or relationships they notice from their experiments.

Using a Computer

To obtain many more estimates, the previous process can be modeled on a computer. The output for four runs each consisting of 15,000 trials is given in Table 4. What do these numbers mean? Considering the probabilities calculated in Table 3, the output from this program would seem to suggest that each of the expected number of rolls to get a new prize is in some way associated with the probability of getting the new prize. With a little guidance from the teacher and the summary of results (Tables 2, 3, 4), students can discover a relationship between the probability of getting a different prize and the expected number of rolls to obtain it. By comparing the values in Table 3 to their experimental results in Table 4, students will see that the expected number of times that they would have to roll the die to get a new prize is very close to the reciprocal of the probability of getting a new prize in each particular case.

With this discovery, some students will be convinced of a pattern that can be used to solve the Cereal Box problem. That is, by summing the average number of rolls for obtaining each successive prize (the reciprocal of the probability), an estimate of the expected value can be obtained:

$$\frac{6}{6}+\frac{6}{5}+\frac{6}{4}+\frac{6}{3}+\frac{6}{2}+\frac{6}{1}=14.7.$$

Some students may want to look for a general proof of their findings, which leads to the following investigation using a theoretical model based on the above Monte Carlo simulations.

A Theoretical Investigation

As all successful problem solvers know, sometimes solving a simpler problem and generalizing to the larger problem is easier (Polya, 1948). Based on the previous experimental results, the following theoretical investigation intends to show that the expected number of rolls or purchases to obtain a different prize after successfully obtaining a particular prize (or set of prizes) is the reciprocal of the probability of that event happening. This finding can then be generalized to the case of the Cereal Box problem. This theoretical investigation would be a great extension for high-school-aged students or for an introductory college statistics course.

To relate to the previous experiments, if the first roll of the die is a 3, how many rolls would be expected before getting something other than 3 (i.e., 1, 2, 4, 5, or 6)? Consider the possible outcomes in attempting to obtain a new prize, along with their respective probabilities. For example, what is the probability of obtaining a different prize on the fourth roll (e.g., 333*,

Table 4
Results From Four Runs of a Monte Carlo Computer Simulation Estimating the Number of Rolls to Obtain a New Prize.

After Obtaining # Prizes	Estimate of Expected Number of Rolls Using 15,000 Trials Per Run			
	Run 1	Run 2	Run 3	Run 4
0	1	1	1	1
1	1.2004	1.1971	1.2055	1.2026
2	1.5025	1.5008	1.4945	1.5009
3	2.0043	1.9909	2.0219	1.9941
4	3.0117	2.9994	2.9889	3.0103
5	5.9853	5.9457	5.9750	6.0346

Table 3
Probabilities of Getting a New Prize, Reciprocals and Decimal Equivalents.

After Obtaining # Prizes	Probability of Getting New "Prize"	Reciprocal of Column 2	Decimal Equivalent of Reciprocal
0	6/6	6/6	1
1	5/6	6/5	1.2
2	4/6	6/4	1.5
3	3/6	6/3	2
4	2/6	6/2	3
5	1/6	6/1	6

where * represents a prize different from 3)? After obtaining the first prize, the probability of obtaining a different prize on any individual subsequent roll is 5/6, and the probability of obtaining another 3 is 1/6. Therefore, the probability of rolling 333* would be

$$\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) = \left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^3.$$

This provides an opportunity to introduce the properties of probabilities associated with a geometric distribution. Table 5 gives the probabilities of obtaining a different prize on the first through seventh rolls. Note, however, that this pattern of probabilities can go on infinitely, because in theory an outcome different from 3 may never occur. Now, using these values, the average number of rolls expected can be calculated by weighting each number of rolls (1 to ∞) with the probability of its occurrence. Therefore, the expected number of rolls to get a new outcome having chosen the first prize is

$$1\left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^0 + 2\left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^1 + 3\left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^2 +$$

$$4\left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^3 + 5\left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^4 + 6\left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^5 + \dots$$

$$= \sum_{k=1}^{\infty} k\left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^{k-1},$$

where $k =$ number of rolls to get a different prize, (1, 2, ..., ∞). Similarly, having obtained two prizes, the expected number of rolls to get a different prize would be

$$1\left(\frac{4}{6}\right)^1\left(\frac{2}{6}\right)^0 + 2\left(\frac{4}{6}\right)^1\left(\frac{2}{6}\right)^1 + 3\left(\frac{4}{6}\right)^1\left(\frac{2}{6}\right)^2 +$$

$$4\left(\frac{4}{6}\right)^1\left(\frac{2}{6}\right)^3 + 5\left(\frac{4}{6}\right)^1\left(\frac{2}{6}\right)^4 + 6\left(\frac{4}{6}\right)^1\left(\frac{2}{6}\right)^5 + \dots$$

$$= \sum_{k=1}^{\infty} k\left(\frac{4}{6}\right)^1\left(\frac{2}{6}\right)^{k-1},$$

where $k =$ number of rolls to get a new prize, (1, 2, ..., ∞). Note that after obtaining two different prizes the probability of obtaining a new prize on any individual subsequent roll is 4/6.

In general, the number of rolls one would expect in order to get a new prize would be

$$\sum_{k=1}^{\infty} k(p)(1-p)^{k-1},$$

Table 5
Probability of Obtaining a New Prize on a Given Roll.

Getting a new prize on the...	First Prize	Possible Sequence of Rolls	Probability of Occurring
First try	3	5	$\left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^0$
Second roll	3	36	$\left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^1$
Third roll	3	334	$\left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^2$
Fourth roll	3	3332	$\left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^3$
Fifth roll	3	33335	$\left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^4$
Sixth roll	3	333331	$\left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^5$
Seventh roll	3	3333334	$\left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^6$
⋮		⋮	⋮

where p is the probability of a new prize and $(1 - p)$ is the probability of one of the previously obtained prizes. For convenience the formula can be written as

$$p\sum_{k=1}^{\infty} k(1-p)^{k-1}.$$

By dropping the constant p, continue the explanation using the simpler formula

$$\sum_{k=1}^{\infty} k(1-p)^{k-1}.$$

Expanding the formula results in the following:

$1(1-p)^0 + 2(1-p)^1 + 3(1-p)^2 + 4(1-p)^3 + 5(1-p)^4 + 6(1-p)^5 + \dots,$

which can be further expanded as follows:

i. $1 + (1-p) + (1-p)^2 + (1-p)^3 + (1-p)^4 + (1-p)^5 + \dots$
ii. $+ (1-p) + (1-p)^2 + (1-p)^3 + (1-p)^4 + (1-p)^5 + \dots$
iii. $+ (1-p)^2 + (1-p)^3 + (1-p)^4 + (1-p)^5 + \dots$
iv. $+ (1-p)^3 + (1-p)^4 + (1-p)^5 + \dots$
v. $+ (1-p)^4 + (1-p)^5 + \dots$
vi. $+ (1-p)^5 + \dots$
vii. $\dots$

$$= \sum_{i=0}^{\infty} 1(1-p)^i + \sum_{i=1}^{\infty} 1(1-p)^i + \sum_{i=2}^{\infty} 1(1-p)^i + \sum_{i=3}^{\infty} 1(1-p)^i$$
$$+ \sum_{i=4}^{\infty} 1(1-p)^i + \sum_{i=5}^{\infty} 1(1-p)^i + \dots$$

$$(i + ii + iii + iv + v + vi + vii)$$

Notice, an interesting thing happens. Each of the separate components of the summation, lines *i* - *vii*, is a geometric series. Recall that a convergent geometric series has the form

$$\sum_{i=0}^{\infty} ar^i = a + ar + ar^2 + ar^3 + \ldots,$$

where $a \neq 0, r < 1,$ and the sum of such a series equals

$$\frac{a}{1-r},$$

where a is the first term in the series and r represents a constant rate. For example, consider line *i*, with $a = 1$ and rate $r = (1 - p)$; the sum is equal to

$$\frac{1}{1-(1-p)}.$$

In *ii*, $a = (1 - p)$ and $r = (1 - p)$, resulting in a sum of

$$\frac{(1-p)}{1-(1-p)}.$$

Similarly for line *iii* with $a = (1 - p)^2$ and rate, $r = (1 - p)$, the resulting sum is

$$\frac{(1-p)^2}{1-(1-p)}, \text{ etc.}$$

Simplifying the denominators of each of these sums gives

$$\sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{1}{p} + \frac{1-p}{p} + \frac{(1-p)^2}{p} + \frac{(1-p)^3}{p} + \frac{(1-p)^4}{p} + \frac{(1-p)^5}{p} + \ldots$$

$$= \frac{1}{p}\left(1 + (1-p) + (1-p)^2 + (1-p)^3 + (1-p)^4 + (1-p)^5 + \ldots \right).$$

Recognizing the part contained within the outer parentheses as yet another geometric series with $a = r$ and $r = (1 - p)$, this equation can be simplified further:

$$\sum_{k=1}^{\infty} k(1-p)^{k-1}$$

$$= \frac{1}{p}\left(\frac{1}{1-(1-p)}\right)$$

$$= \frac{1}{p}\left(\frac{1}{p}\right) = \frac{1}{p^2}$$

Recall the first two equations,

$$\sum_{k=1}^{\infty} k(p)(1-p)^{k-1} = p\sum_{k=1}^{\infty} k(1-p)^{k-1}$$

Substituting

$$\frac{1}{p^2}$$

for

$$\sum_{k=1}^{\infty} k(1-p)^{k-1}$$

gives

$$p\sum_{k=1}^{\infty} k(1-p)^{k-1} = p\left(\frac{1}{p^2}\right) = \frac{1}{p}.$$

Recall that p is the probability of getting a new prize, given that a certain number of prizes have been obtained. Therefore, the expected number of rolls to obtain a different outcome is the reciprocal of the probability of rolling a new outcome.

A more elegant proof attributed to Mostellar (1965) is included here for the more advanced investigator. Consider a series of repeated independent experiments where each experiment is a success, with probability p, or failure, with probability $1 - p$ $(0 < p < 1)$. Let $E[N]$ be the expected number of experiments until the first success occurs. Consider the first experiment. If it is a success, then $E[N] = 1$ and the experiments cease. If the first experiment is a failure, independence gives that $E[N]$ for the second experiment on is the same as the original (unknown) $E[N]$, and thus the expected number of experiments required is $1 + E[N]$. Therefore, $E[N] = 1 \cdot p + (1 - p)(1 + E[N])$. Solving this linear equation for $E[N]$ gives that, if the expected value exists, then

$$E[N] = \frac{1}{p}.$$

Now, generalizing to the original Cereal Box problem the expected number of rolls before getting all six outcomes would be

$$\frac{6}{6} + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} = 14.7$$

and, in general,

$$\frac{N}{N} + \frac{N}{N-1} + \ldots + \frac{N}{3} + \frac{N}{2} + \frac{N}{1},$$

where N is the number of free prizes available. Now, given any number of prizes, the expected number of boxes can be calculated using the above formula. For example, suppose that a cereal company is advertising 10 prizes. The expected number of purchases would then be

$$\frac{10}{10}+\frac{10}{9}+\frac{10}{8}+\frac{10}{7}+\frac{10}{6}+\frac{10}{5}+$$

$$\frac{10}{4}+\frac{10}{3}+\frac{10}{2}+\frac{10}{1}\equiv 38.2897.$$

Additional extensions might include having students use Euler's approximation for the partial sum of the harmonic series:

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N}$$

$$=\log_e N+\frac{1}{2}N+0.57721\ldots,$$

to approximate the expected value for a large number of prizes (e.g., see Mosteller, 1965).

Conclusion

The Cereal Box problem provides a challenge for all levels of mathematics students. It serves as a great introduction to situations of uncertainty and the ideas associated with expected value. The problem opens up all sorts of opportunities for conducting simulation experiments as well as theoretical modeling. Although the problem is very complex, by using Monte Carlo methods, even early middle-school aged students can make reasonable estimates about the number of cereal boxes one would have to buy. Motivated by the experiments, more advanced students can discover and investigate theoretical models that can encompass many new mathematical ideas, such as the properties of the geometric series. These sorts of real-world investigations involve students in a context rich in mathematical inquiry.

References

Konald, C. (1991). Understanding students' beliefs about probability. In E. V. Glaserfeld (Ed.), *Radical constructivism in mathematics education* (pp. 139-156). Dordrecht: Kluwer.

Konald, C. (1994). Teaching probability through modeling real-world problems. *Mathematics Teacher, 87*(4), 233-235.

Lappan, G., & Winter, M. J. (1980). Probability simulation in middle school. *Mathematics Teacher, 73*(6), 446-449.

Litwiller, B. H., & Duncan, D. R. (1992). Prizes in cereal boxes: An application of probability. *School Science and Mathematics, 92*(4), 193-195.

Maull, W., & Berry, J. (1997). Modeling the coupon collector's problem. *Teaching Statistics, 19*(2), 43-46.

Mosteller, F. (1965). *Fifty challenging problems in probability with solutions.* Reading, MA: Addison-Wesley.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

Polya, G. (1948). *How to solve it.* Princeton, NJ: Princeton University Press.

Richbart, L., & Richbart, C. (1992). Trading Alfs for Barts: A simulation. *Arithmetic Teacher, 40*(2), 112-114.

Riddiough, R., & McColl, J. (1996). How long will it take to build Jack's house? *Teaching Statistics, 18*(3), 82-85.

Shulte, A. P., & Smart, J. R. (1981). *Teaching statistics and probability.* Reston, VA: National Council of Teachers of Mathematics.

Simon, J., Weidenfeld, D., Bruce, P., & Puig, C. (1995). *Resampling stats* (Version 4.0 for Windows 95). Arlington, VA: Resampling Stats.

Sobol′, I. M. (1994). *A primer for the Monte Carlo method.* Ann Arbor: CRC Press.

Travers, K. J. (1981). Using Monte Carlo methods to teach probability and statistics. In A. P. Shulte & J. R. Smart (Eds.), *Teaching statistics and probability.* Reston, VA: National Council of Teachers of Mathematics.

Travers, K. J., & Gray, K. G. (1981). The Monte Carlo method: A fresh approach to teaching probabilistic concepts. *Mathematics Teacher, 74*(5), 327-334.

Travers, K. J., Stout, W. F., Swift, J. H., & Sextro, J. (1985). *Using statistics.* Menlo Park, CA: Addison-Wesley.

Zawojewski, J. S. (1991). *Dealing with chance. Addenda series, Grades 5-8.* Reston, VA: National Council of Teachers of Mathematics.

Author Note: The author would like to thank Kenneth J. Travers for first introducing him to the Cereal Box problem and making comments and suggestions on a previous draft of this work. The author would also like to thank Tim Hendrix, Debbie Reese, George Reese, Michelle Wilkins, and the reviewers for their helpful suggestions for improving the article. In particular, the author would like to thank one reviewer for offering a more elegant proof attributed to Mostellar (1965) which was also included in the article.

Correspondence concerning this article should be addressed to Jesse L. M. Wilkins, Department of Teaching and Learning, Virginia Polytechnic Institute and State University, 300-C War Memorial Hall, Blacksburg, VA 24061-0313. Electronic mail may be sent via Internet to wilkins@vt.edu.