A Handbook for
Carrying out a Mathematics Department Self-Assessment and Indicators Study

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Foreword: A roadmap to this handbook

This manual is designed to be of assistance to individuals in a mathematics department upon whom the responsibility has fallen to gather data on the status of the teaching and learning of mathematics. This document resulted from the experiences of persons in three very different post-secondary institutions: a community college, a comprehensive state university and a research university.

Section I outlines the overall objectives of the indicators project and identifies the major issues that the project addresses.

This Handbook is divided into the following major sections:

- **Section II** takes up each of the basic issues in more detail, especially the kinds of information that may be useful to address each of them. It also lists the indicators that are associated with each of these issues.

  In this project, information was provided through surveys. Data for issues 1-4 were provided by department administrators (head, associate head, etc) or a designee. Data for issues 5-7 were obtained from the instructional staff (those delivering instruction at the first two years). For issues 8-10, direct input from the students was obtained.

  As is noted below, supplementary data may be obtained through optional activities such as textbook analyses and focus groups. For details on these activities, please see Appendix A. __.

- **Section III** discusses the three survey instruments that were used to collect the desired information from the department, the instructional staff and from the students, respectively.

- **Section IV** presents recommendations on how to actually carry out the study (e.g., planning; data gathering; data processing; reporting.)

Part II
Part III
Part IV

Appendix A:
Appendix B:
Appendix C:
I. Introduction

Undergraduate mathematics teaching today is taking place in an environment of rapid, often dramatic, change. These changes include:

- the demographics and mathematical background of students taking college mathematics courses;
- the mathematical needs and expectations of partner disciplines, graduate programs, prospective employers;
- the kinds of classroom practices that are generally considered “exemplary”; and
- the increasing use of technology to enhance the teaching and learning of mathematics.

Moreover, we live in a day and age of increased emphasis on “accountability”—accountability to central administration, funding agencies, government, the public, and (not least!) our students. In view of this double climate of change and accountability, it would seem helpful for a mathematics department to periodically take stock of its programs. Perhaps the reasons for doing a departmental self-evaluation are most succinctly and eloquently summarized in Lincoln’s words in the “House Divided” speech, delivered in June 1858:

“If we knew where we are, and whither we are tending, we could better judge what to do and how to do it.”

This handbook contains a set of recommendations on how to plan, carry out, and make use of a departmental self-evaluation or “stock taking.” The ideas presented here are based on what was learned in the NSF-funded project, ‘Developing Statistical Indicators to Monitor the Condition of Undergraduate Mathematics Education: A Feasibility Study’. In this project, three mathematics departments each carried out an extensive self-study of its lower division (first two years) mathematics programs, with a view to learning how to effectively carry out such an activity. Questions asked included: What kinds of data should be collected? How should one go about gathering such information? And, finally, how can the information gathered be most effectively used?

The three mathematics departments participating in the Indicators Project were:

- Comprehensive state university (CU)
  CU is a part of a major state campus system. It is relatively new, having opened its doors to several hundred upper division students in fall, 1990, and enrolled its first freshman class (about 1,000 students) in fall 1995. At the time of the Indicators study, over five thousand students were in attendance.

- Midwestern community college (CC)
  CC is a comprehensive community college dedicated to providing programs and services of high quality to its students and committed to continuous improvement, to academic achievement and its documentation, and to the concept of shared governance. In fall 1998 there were 10,108 students enrolled, 8,564 credit and 1,544 non-credit. Of these students, 41% were full-time, 57% were female. The average age of all students was 28 years. Seventy-two percent of the students were freshmen, the remainder were sophomores. The faculty-student ratio is approximately 1:19.

- Research I university (RU)
  RU is a major, public, university enrolling approximately 27,000 undergraduates and 6,000 graduate students in the fall of 1998. In the academic year 1998-99 over 9,000 degrees (undergraduate and graduate) were awarded. Of the undergraduate students, 47% were female. In terms of race/ethnicity, 7% of the undergraduates were Black, 5% Hispanic and .2% Native American. The total number of full-time equivalent faculty was 2,756. The graduate assistants, numbering 5,491 in that year, provide an important part of the instructional support, especially at the undergraduate levels.
The choice of three such very different kinds of institutions for the Indicators Project was deliberate. It allowed us to see how such a self-study plays out in three very different kinds of settings. And, indeed, each of the three departments did in fact produce a somewhat different “departmental profile”.

Nevertheless, there was also a large core of common issues and concerns across the three institutions. To a great extent, the following set of issues embraces this common core.

**A fundamental question: What does the department want to know about itself, and why?**

Before your department embarks on a project such as this—with all of the data collection and analysis activities that are entail, it is important to have a wide-ranging that includes the stakeholders in the department about the major goals this self-study. At first blush, it might seem that the more information you collect, the better off you will be—that is to say, the best snapshot of your department will be the one with the highest resolution (i.e. the greatest number of data points).

But this idea is misleading on several counts, in large part, because gathering information does not come for free. The price to be paid includes some or all of the following factors:

- Gathering and processing data costs time and money.
- Faculty (and students) have their limits as to how much time and effort they will be willing to put into filling out questionnaires, keeping records of what was taught, being observed, etc. Ask too much and/or too often, and you will find that initial cooperation may turn into non-cooperation and even resentment (or retirement!).
- Even if you WERE to collect reams of data, too much can easily lead to “information overload.” There is so much to think about that one quickly feels overwhelmed—even paralyzed—trying to make sense of it all.

Thus, before you get started, it pays to think hard and carefully about what kinds of data you want to collect, and why. Two preliminary questions we suggest that you address at the very outset are:

- What does the department think are the important questions and issues it faces?
- What kinds of data and information would best help shed light on those questions and issues?

In the following pages, we list and discuss some of the issues that we believe are important for a mathematics department to address. These were distilled from a series of panel meetings with mathematicians and mathematics educators in a prior study, in which a conceptual framework was developed for identifying key issues in the teaching and learning of mathematics in the first two years of college.1

Since every mathematics department is unique and special in some ways, it is quite likely that you will want to add to (as well as delete from) our list of questions. However, please keep in mind that the ‘Indicators Project’ is built upon a foundation of key issues and associated indicators in undergraduate mathematics education. Inasmuch as you depart from what is here, you may be forfeiting some opportunities for comparing data from your department to what was found in our project. In any case, what is of paramount importance is that you do all you can to make sure that all the various stakeholders are duly consulted, and brought on board from the very outset of the study.

The following is a listing of the issues that were addressed in the current project. These may be helpful as you plan your own data collection exercise.

**Issue 0: Department demographics**
Descriptive information about the institution and the department: student enrollment; size of mathematics faculty; levels of instruction, etc.

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1 See Volume I (Main Report) Appendix A for a list of these persons.
**Issue 1: Department goals and priorities**
What does the department think of as its primary *mission*? In particular, what are the relative emphases placed on instruction, research and public service?

**Issue 2: Program maintenance and monitoring**
What provisions are there in the department for review and revision of programs and course offerings?

**Issue 3: Professional development and instructional staff support**
What opportunities for staff development are supported and/or encouraged by the department?

**Issue 4: Partner disciplines**
What are the various audiences (including teacher education) served by the department?

**Issue 5: Instructional strategies**
What instructional strategies are used? Especially, what use is made of interactive teaching strategies—those intended to encourage the students to be active classroom participants?

**Issue 6: Classroom uses of technology**
What kinds of technology (calculators, computer software, the Internet, etc.) are used for instruction?

**Issue 7: Assessment**
What use is made of assessment (quizzes, homework, examinations) to inform the instructor about student progress?

**Issue 8: Student retention**
What percent of students continue in mathematics or mathematics-based programs? As a related issue to retention, what are student views about the mathematics being taught? What opinions do students have about the utility of the mathematics they have learned—in later courses, or in their careers?

**Issue 9: The mathematics department as a community of learners**
To what extent are students involved in the life of the department?

**Issue 10: Diversity**
To what extent does student enrollment in mathematics courses reflect the campus population as a whole (say, in terms of gender and ethnicity)?

Note: While these issues are intended to carve things into *roughly* mutually exclusive categories, the categories aren’t *strictly* mutually exclusive. For example, in practice, the kinds of teaching methods being used, the varieties of technology implemented and the ways that student progress is assessed can sometimes be pretty hard to disentangle.
II. Some basic questions (issues)

**Issue 0: Department demographics**

To begin to understand a department, one must get a feel for its composition. Thus, among the first kinds of data to collect should be those that give a demographic profile of the department. Examples of such demographic information include:

- **Instructional staff**
  This includes such items as the staff’s overall size as well as its composition by rank, professional background, gender, ethnicity, and interests.

- **Student body**
  The make up of the student body, or student demographics, might be portrayed by data such as: How students are characterized by gender, age, ethnicity, etc.? Also, how they are distributed over various categories such as: mathematics majors; majors in various partner disciplines; part-time students; pre-service teachers; adults returning back to school? And it is also important to have data on student enrolment in the various courses offerings.

**Selected indicators:**
The following indicators have been selected for study, with data provided by each of the three pilot sites.

<table>
<thead>
<tr>
<th>Issue 0: Department Demographics</th>
<th>Key Indicator 0.0: Characteristics of instructional staff and student body</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1 Instructional staff by tenure status and gender</td>
<td>0.2 Instructional staff by tenure status and race/ethnicity</td>
</tr>
<tr>
<td>0.3 Student characteristics by enrollments in departmental course offerings</td>
<td>0.4 Student admissions policies (mathematics requirements for admission)</td>
</tr>
</tbody>
</table>

**Issue 1: Department goals and priorities**

As was noted earlier, it seems clear that any departmental self-assessment should be based on the department’s idea of its primary mission(s). If there is not already a formal mission statement, this would be a good time to create one. It is essential that a departmental mission statement reflects a wide consensus, so it needs to be crafted with comprehensive input from the departmental faculty, as well as consultations with central administration, partner discipline departments (engineering, physics, economics, education, etc.) as well as, possibly, adjunct instructors, students and alumni.

Mission statements are sometimes dismissed as collections of empty platitudes. But this certainly need not be the case: They can provide a focus for a department in terms of priorities and resource allocation.

Mission statements will differ from one department and institution to another. For example:

A mathematics department in a **comprehensive state university** might have a significant mission of: “preparing K-12 teachers of mathematics.” In that case, appropriate questions to ask in a self-assessment study might include: Are there mechanisms in place to facilitate communication and cooperation between the mathematics department and the institution’s school of education? How well are these intra-institutional mechanisms working, e.g., are the mathematics and education courses well coordinated? Also, is there good articulation between the department and the schools, e.g., does the department get useful feedback about its pre-service courses for teachers from the schools in which its graduates are placed?

In a **community college** setting, a departmental mission might be: “to provide adults in the community with opportunities to learn mathematics for job or career related purposes.” In such a case, one would probably want to address questions like: How many adults in the community actually avail themselves of the opportunities provided by the department? Are the department’s courses offered at times and in
locations convenient for the targeted audience of students? Do students (actual and prospective) feel that these courses meet their needs? Is there coordination with, and input from, local employers?

On the other hand, in a major research university, one of the goals of undergraduate mathematics education might be: “to identify and groom prospective research mathematicians.” In such a setting, important questions to ask might include: Are there outreach programs to attract high achieving high school students to enroll in the department’s courses? Are there mentoring programs for advanced placement undergraduate majors, including, e.g., opportunities for students to participate in the research activities of individual members of the instructional staff? What proportion of the department’s undergraduate mathematics majors go on to graduate studies in mathematics or mathematics-related disciplines?

In order to determine the extent to which departmental policies and practices reflect and support its mission statement, you might begin by polling the instructional staff, asking such questions as:

- Are you aware of the existence of a departmental mission statement?
- Do you agree with the current mission statement?
- In what ways would you personally like to see the mission statement modified?
- Do you think the department is currently meeting the goals, as expressed in its mission statement?
- How important is the mission statement to you in your everyday work, especially teaching?

If you find that a significant number of the instructional staff are unfamiliar with the department’s mission statement, or disagree with it, or find it irrelevant, or believe the department fails in large measure to meet its stated aims, that would surely be food for thought, and probably a spur to some remedial action.

Of course, questions like the ones above address only faculty opinions about the department’s mission. In this data collection project, you will also want to look at the extent to which actual department policies and practices are in line with the mission statement. Again, if the study turns out to reveal a significant divergence between the mission statement and actual practice, then something is amiss. In such a case, you might want to rethink and/or modify the mission statement, or, perhaps you will need to readjust departmental policies and practices to conform more closely to stated departmental goals— or, possibly, both.

**Selected study indicators:**

<table>
<thead>
<tr>
<th>Issue 1: Department goals and priorities</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Key Indicator 1.0: Departmental emphasis on undergraduate instruction</strong></td>
</tr>
<tr>
<td>1.1 FTEs committed to undergraduate instruction; graduate instruction; service and research</td>
</tr>
<tr>
<td>1.2 Instructional staff by tenure status teaching lower-division courses</td>
</tr>
<tr>
<td>1.3 Instruction takes place in flexible settings</td>
</tr>
<tr>
<td>1.4 Facilities are available to enable best possible teaching methods</td>
</tr>
</tbody>
</table>

**Issue 2: Program maintenance and monitoring**

What provisions are there in the department for review and revision of course offerings?

In addressing this issue, it is helpful to begin by putting together a comprehensive overview of existing departmental offerings, including:

- The various programs the department offers (e.g., mathematics; statistics; combined mathematics/computer science; teacher preparation; actuarial science; etc.);
- The degrees and/or certificates offered;
- The prerequisites and requirements for the department’s various programs;
- A complete list of courses (sorted into categories such as: required for majors, elective, service, etc.) with prerequisites and a brief description of each course;
- The instructional materials used in each course (e.g., textbooks, software packages; etc.).
You will probably also want to see how your department’s courses, programs, and curricula relate to recommendations of various national reports: e.g., in the case of a community college, Crossroads in Mathematics: Standards for Introductory College Mathematics Before Calculus (Memphis, TN: American Mathematical Association of Two-Year Colleges, 1995); in the case of four year institutions, for example, Shaping the Future, New Expectations for Undergraduate Education in Science, Mathematics, Engineering and Technology (Arlington, VA: National Science Foundation Publication 96-139, 1996); for teacher education programs, CBMS Guidelines for the Mathematical Education of Teachers. Draft report available at www.maa.org/cbms

For examples of ways to study these questions in more depth, see the optional data collection activities in Appendix A.

Selected study indicators:

<table>
<thead>
<tr>
<th>Issue 2: Program maintenance and monitoring</th>
<th>Key Indicator 2.0: Departmental provision for review and revision of course goals and content.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>There is a written statement of department goals</td>
</tr>
<tr>
<td>2.2</td>
<td>A departmental syllabus exists for each course</td>
</tr>
<tr>
<td>2.3</td>
<td>Provisions are made in the department for the revision of courses</td>
</tr>
<tr>
<td>2.4</td>
<td>Course goals and content have changed within the past five years</td>
</tr>
<tr>
<td>2.5</td>
<td>Changes in course goals and content reflect recommendations of national reports</td>
</tr>
<tr>
<td>2.6</td>
<td>The department keeps abreast of the changing course needs of its students</td>
</tr>
<tr>
<td>2.7</td>
<td>The concerns the department in choosing a text</td>
</tr>
</tbody>
</table>
Issue 3: Professional development and instructional staff support

This issue focuses on the extent to which provisions are made for the continuing professional development of the instructional staff. In addressing this issue, the following kinds of information would be useful as background:

- *Academic preparation and experience*: e.g., degrees earned; special areas of interest and expertise; years of teaching experience.

- *Professional life*: e.g., Balance of faculty time devoted to research, teaching, and public service; membership and leadership positions in professional societies; publications in professional journals; attendance and presentations at professional meetings.

- *Technology*: Experience, expertise, and attitudes about the use of technology in the teaching of mathematics.

**Selected study indicators:**

<table>
<thead>
<tr>
<th>Key indicator 3.0: Departmental support of instructional staff</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1 The department supports and encourages the professional development of its instructional staff in the teaching and learning of mathematics</td>
</tr>
<tr>
<td>3.2 The department supports and encourages the professional development of its instructional staff in the knowledge of and ability to do mathematics</td>
</tr>
<tr>
<td>3.3 The instructional staff is using instructional technologies in the classroom</td>
</tr>
<tr>
<td>3.4 The instructional staff is using innovative instructional approaches in the classroom</td>
</tr>
<tr>
<td>3.5 The instructional staff participates in professional associations</td>
</tr>
</tbody>
</table>

Issue 4: Partner disciplines

This indicator focuses on the degree to which the goals of the department address the needs and desires of a wide range of users of the mathematical sciences (various 'partner disciplines' across the campus—for example, engineering, the sciences, economics and finance, statistics, mathematics education, liberal arts and sciences.

**Selected study indicator:**

<table>
<thead>
<tr>
<th>Key Indicator 4.0: Departmental communication with partner disciplines about student needs and program content</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1 The goals of the department speak to the needs and desires of a wide range of users of the mathematical sciences (that is, the ‘partner disciplines’)</td>
</tr>
</tbody>
</table>
Issue 5: Instructional strategies

Important questions relating to classroom practices include:

- What is the role of lectures, whole class discussions, going over homework, small group work, etc.?
- To what extent are strategies used by instructors to encourage active student participation in the ongoing work rather than being a passive member of the class?
- How is class time distributed among the various kinds of activities?

Selected study indicators:

<table>
<thead>
<tr>
<th>Issue 5: Instructional strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Key indicator 5.0: Use of interactive teaching strategies</strong></td>
</tr>
<tr>
<td>5.1 Instructors use a variety of teaching strategies</td>
</tr>
<tr>
<td>5.2 Instructors use a variety of interactive (non-lecture) teaching strategies</td>
</tr>
<tr>
<td>5.3 Instructors use strategies to promote student interaction</td>
</tr>
<tr>
<td>5.4 Instructors use a variety of mathematical representations while teaching</td>
</tr>
<tr>
<td>5.5 Instructors promote active engagement with mathematical content</td>
</tr>
<tr>
<td>5.6 Instructors are available to students outside of class</td>
</tr>
</tbody>
</table>

Issue 6: Classroom uses of technology

In recent years, technology has profoundly affected the way that mathematics research is done, as well as the ways mathematics is applied in the ‘real’ world, for example: vastly enhanced possibilities for data manipulation and visualization. Similarly, technology has also affected the ways that mathematics can be taught and learned: computational, symbolic computational, and graphing technology are now readily and widely available, both for instructional use and for student hands-on experiences.

This recent increase in the power, prevalence and accessibility of technology raises questions about how such new technologies can and do interact with instruction. Important questions include:

- What kinds of technologies are used for teaching in various courses?
- How are these tools used?
- What instructional objectives are promoted by the use of these technologies (that is, what is the ‘value added’ by such innovative tools?)

Selected study indicators:

<table>
<thead>
<tr>
<th>Issue 6: Classroom uses of technology</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Key indicator 6.0: Use of technology in the classroom</strong></td>
</tr>
<tr>
<td>6.1 Classrooms are equipped for using technology in instruction</td>
</tr>
<tr>
<td>6.2 Courses required by the department to use technology for instruction</td>
</tr>
<tr>
<td>6.3 Department provides support for technology for mathematics instruction</td>
</tr>
<tr>
<td>6.4 Department provides support for technology for mathematics research</td>
</tr>
<tr>
<td>6.5 Instructors use calculators in teaching</td>
</tr>
<tr>
<td>6.6 Instructors use technology (other than calculators) in teaching</td>
</tr>
<tr>
<td>6.7 Technology is used in teaching a variety of mathematics courses</td>
</tr>
<tr>
<td>6.8 Instructional staff considers itself proficient in using calculators or computers for teaching purposes</td>
</tr>
</tbody>
</table>
Issue 7: Assessment

Assessment methods traditionally have used paper and pencil tests, quizzes, and/or homework. More recently, alternative forms of assessment have been suggested: projects, writing assignments, collections of student work, etc. There is also ongoing discussion about whether this assessment should be used merely to grade students (summative evaluation) or also to help students learn more effectively (formative evaluation). Formative evaluation can inform and direct teaching and learning by helping diagnose what students are doing and how they might refocus their efforts. A number of questions follow from the above issues: What assessment methods are actually used in various departmental courses? How frequently are these methods used and/or emphasized? Which, if any, are used for feedback and "formative" assessment of student progress? Which are used in final "summative" grading?

Selected study indicators:

<table>
<thead>
<tr>
<th>Issue 7: Assessment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Key Indicator 7.0: Use of assessment methods</td>
</tr>
<tr>
<td>7.1  Instructors use a variety of assessment methods</td>
</tr>
<tr>
<td>7.2  Instructors seek student feedback to monitor progress</td>
</tr>
<tr>
<td>7.3  Instructors use a variety of criteria in determining final grades</td>
</tr>
<tr>
<td>7.4  Instructors assess core student proficiencies using common items</td>
</tr>
</tbody>
</table>

Issue 8: Student retention

A perennial concern for college mathematics instruction is the high failure and drop out rate in many courses. The issues are varied. What are the failure and withdrawal rates for key lower division mathematics courses? Do they affect certain kinds of students more than others (e.g., minorities, those returning to school, part-time students, etc.)? Are there specific mechanisms in place designed to increase retention and success? How do students perceive mathematics courses -- as filters or hurdles, or as something more positive? Do students believe that instruction in their courses is effective? Do they see the content of their courses related to their anticipated fields of study and work? Do they think that the department and their instructors as concerned about them and are actively involved in helping them succeed?

Selected study indicators:

<table>
<thead>
<tr>
<th>Issue 8: Student retention</th>
</tr>
</thead>
<tbody>
<tr>
<td>Key Indicator 8.0: Student intention to continue in the study of mathematics</td>
</tr>
<tr>
<td>8.1  Students feel that the instructor is aware of mathematical needs of their major field of study</td>
</tr>
<tr>
<td>8.2  Students believe that the content of the course they have just completed will be useful in their future</td>
</tr>
<tr>
<td>8.3  Students look forward to taking more mathematics</td>
</tr>
<tr>
<td>8.4  Proportionate numbers of women and ethnic minorities intend to continue in the study of mathematics</td>
</tr>
</tbody>
</table>
Issue 9: The mathematics department as a community of learners

Mathematics students may feel disconnected from the ongoing life of the department, concerned only with the specific course(s) in which they are enrolled—course(s) in which they may often feel that they are only one more anonymous ‘warm body’. As Ewell (199_) has noted,

Researchers on college student settings and behaviors have over the past twenty years consistently identified a particular set of attributes of the institutional environment that appear to be associated with high involvement and achievement (Astin 1993, Pace 1990, Pascarella and Terenzini 1991). Among these measures are: frequent opportunities for out-of-class contact between faculty and students; departmental (and/or institutional) policies and procedures that treat students as "partners" or "valued customers." and the presence and use of specific mechanisms for regularly identifying student needs and difficulties. (Volume II, p. __ )

Selected study indicators:

<table>
<thead>
<tr>
<th>Key Indicator 9.0: Student participation in the life of the department</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.1 Students take part in supplementary (non-class) mathematical support services activities</td>
</tr>
<tr>
<td>9.2 Students take part in supplementary (non-class) mathematical activities (e.g. special lectures, colloquia, math clubs, etc.)</td>
</tr>
<tr>
<td>9.3 Students feel that technology (calculators/and or computers) is helpful in learning mathematics</td>
</tr>
<tr>
<td>9.4 Students take part in social activities of the department</td>
</tr>
</tbody>
</table>

Issue 10: Diversity

College mathematics has often been portrayed as being the preserve of white middle and upper class males. Various mandates and programs have been suggested to try to remedy this situation (reference, for example, the NSF initiatives for minority participation in the mathematical sciences in their Human Resources Development Programs.) As a recent (2000) program announcement from NSF has noted:

The U.S. risks losing the scientific, economic and human resource advantages it now enjoys without an IT workforce that is large enough to meet both the public and private sectors' growing demand, and that is adept at using and producing information technologies. In this respect, the under-representation of women and minorities in computer science and engineering (CS&E) is a serious national problem. There is agreement among some of the nation's leading researchers and scientists that systematic research efforts are needed to address this problem. CISE is continuing its research program on the IT workforce (ITWF) and will support a broad set of scientific research studies focused on the under-representation of women and minorities in the IT workforce. (reference: http://www.nsf.gov/cgi-bin/getpub?nsf0133)

Selected study indicators:

<table>
<thead>
<tr>
<th>Key Indicator 10.0: Proportional representation by age, gender, and race of students in mathematics courses</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.1 (Lower division) student enrollment in the department shows gender diversity</td>
</tr>
<tr>
<td>10.2 (Lower division) student enrollment in the department shows racial/ethnic diversity</td>
</tr>
</tbody>
</table>
III. Data collection: Sources and instruments

**Surveys** provide the major source of data for this project is that of surveys. Three such surveys were developed as noted below.

- **Department profile:**
  Summarizes the basic “demographics” of the department, including, e.g., departmental course offerings and programs; instructional staff (identified by rank, full time vs. part time, etc); the number of mathematics majors; the number of classes; student enrollments in these courses; support facilities (e.g., tutoring and computer laboratories; etc). This form should be completed by the department executive officer, or someone designate by him/her.

- **Instructional staff survey:**
  Provides data from individual instructors concerning their academic background, professional activities and interests, attitudes to mathematics, attitudes to teaching, technology, etc. The survey has two parts: Part One: General background information; Part Two: Data referring specifically to the course(s) being surveyed.

- **Student surveys:**
  These ask about the students’ mathematical backgrounds (e.g., from high school), career plans, attitudes towards mathematics, individual courses and their reactions to individual courses as well as to the mathematics department as a whole. There are two versions: a short version that is intended to be used at the beginning of the semester and a longer one for use at the end of the semester. Some items (for example, attitudes toward technology and toward mathematics) appear on both surveys in order to permit assessing attitudinal change.
IV. Some recommended steps to follow

At a minimum, a study of this type will require three semesters to complete—one full semester each of:

- planning;
- data gathering; and
- data analysis, followed by interpretation, dissemination, reflection and discussion of implications

Special note: Sample memoranda
As you read through these guidelines, we suggest that you also take a look at the various memoranda that have been provided as documents that might be useful to you as you communicate with colleagues, administrators or students about the goals and requirements of this data collection activity.

Semester I: Planning

Involve all stakeholders: At the very outset, you should notify all the stakeholders of your intention to carry out a self-evaluation study of the department, and throughout the planning phase, you should solicit their ideas and cooperation. This in turn means that you should invite the following to at least some of your planning sessions:

- the departmental faculty (including, where relevant, part-time faculty and graduate teaching assistants);
- the executive officers of the department;
- representatives from central administration;
- representatives from partner departments;
- representatives from the appropriate support offices (e.g., data processing units and central record keeping);

In addition, it may also be useful to involve:

- student representatives.

To continue to keep the stakeholders informed throughout the course of the study, consider:

- reserving a spot from time to time at department meetings for progress reports;
- setting up an (electronic) mailing list for periodic updates; and
- setting up a Web page for the study.

§ See Sample memo: B-1-1 (Appendix B, Semester 1, Memo 1) ‘Setting the stage for the Indicators Project’

2. Advisory Board: An advisory board is important to help you keep the study on track. For efficiency, it’s a good idea to keep this board quite small, i.e., it probably suffices to have just three or four people on the board: one member from a partner discipline, and two or three members from outside the institution. The advisory board should meet at least once during each of the three phases of the project: planning, data gathering, and data analysis. But it doesn’t hurt to have the board meet more often if you can persuade the members to do that. Prior to each meeting, you should distribute to the board an agenda of items and issues for which you wish the board’s input.

Even between meetings, you should keep the board well informed of the progress of the study. Also, consider seeking their advice on a variety of questions relating to the conduct of the study, e.g., reviewing the design of your survey instruments; looking at data summaries; etc. Here, as so often, you need to steer a careful course between, on the one hand, keeping your advisory board informed, on the other hand, overwhelming them with questions and data.

3. Privacy issues: Throughout the study, you need to make sure that your procedures protect the rights and privacy of your “subjects,” especially the students in your study. If your institution has an office that deals expressly with such issues, make sure early on to contact this office to get their approval for your questionnaires and all other data-gathering procedures, including access to records from the institution’s
central database for (for example) student records (such as grades). If any federal funds are involved in
your project, you need to refer to the publication: “Federal Policy for the Protection of Human Subjects.” A
useful web site dealing with these kinds of issues is: www.uiuc.edu/unit/vcres/irb.

4. Surveys: As already noted, the study entails using surveys of department representatives, instructors and
students.

Department representative:
http://www.mste.uiuc.edu/indicators/department/DeptProfileInfo.html

Instructors:
http://mstemac4.ed.uiuc.edu:591/Indicators/FacSurveyPt1online.html

Students (beginning of semester):
http://mstemac4.ed.uiuc.edu:591/Indicators/presurv.html

Students (end of semester):

To save yourself work, we strongly encourage you to use these surveys. Moreover, if several colleges and
universities use the same instruments, this can provide a basis for comparison between institutions that
could prove useful on a regional, or even a national, basis. However, to address issues unique to your own
institution or department, you may want to add survey items of your own, or even to create additional
instruments, such as a survey that collects student data at mid-term.

If you decide to supplement the Indicators Study instruments by adding your own items, be sure to have
them critiqued by colleagues, as well as to try them out with small groups of respondents (instructors or
students). This will help avoid technical glitches, such as the amount of time needed to complete the survey
and (most important!) possible misinterpretation of wording of the items. (Remember: if there is any way
in which a question can be misread, it is highly likely that someone will find that way!)

Anonymity of responses is important. Instructors need to be assured that the results will NOT be used in
any way to evaluate them personally as teachers. Students need to be assured that the results will not affect
their grades. On the other hand, you likely will want to be able to link instructors with courses and students.
It is also useful to be able to match up student responses to the pre- and post- surveys in order to detect
possible changes in attitudes (for example, toward mathematics or the utility of technology in a course).
One way to preserve anonymity is to use create codes that identify instructors and students, but to keep
confident the roster that links names to codes.

As you prepare the survey instruments, it is essential to solicit early and continuing input from the data
processing people. For example, you should discuss with them the format of the surveys. You might want
to consider getting responses on-line. If you decide to do so, it is crucial that you try out this mode of data
collection well in advance in order to spot potential bugs in the system.

As you think about collecting data, keep in mind that entering masses of data manually into a computer can
be very expensive and time consuming. For multiple choice items on the student surveys, consider using
“scan sheets” that can be read directly into the computer (This doesn’t mean you shouldn’t also ask open-
ended questions, which can be a source of very valuable information!)

Semester II: Data Collection

It is possible to complete the data-gathering phase in just one semester. However, if you can spare the time
and resources, there are good reasons for extending the data gathering over a period of two consecutive
semesters. Advantages derived from essentially repeating the study in two consecutive semesters may
include the following:
• You get a chance to make some corrections in data gathering procedures that may have gone awry the first time around.
• You also are provided with a reliability check (e.g., to ascertain whether the data from the first and second go-around are reasonably consistent.) On the other hand, it may also reveal that different semesters in the annual school cycle really are different (for example, students taking Calculus I in the fall semester are demographically different from those taking Calculus I in the spring.

Here is an outline of a time line for the data-gathering semester (You may also wish to refer to the sample memoranda provided in Appendix B, as note):

1. The initial student questionnaire should be passed out at the very beginning of the semester, preferably on Day 1. One has a choice of whether to ask students to complete these in class, or to take them home. The advantage of the former is that you can be sure of getting the completed questionnaires back, then and there. The advantages of the latter are: (a) that you’re not taking up valuable class time, and (b) that you may get more thoughtful responses. If you decide on take-home questionnaires, it is important to do all you can to have them returned within one week. § See memos B-2-1, B-2-2

2. Distribute the final student questionnaires during the last two weeks of class. Remember that the end of the semester is a busy time for both students and instructors, so don’t wait till the very last day. If you decide on take-home (rather than, in-class) questionnaires, you need to make doubly sure that they get back to you. Remember: after the end of the semester the students will scatter to the four winds, and you may have real trouble getting anything back (from either student or instructor!). § See memo B-2-3

3. The instructor survey should be completed toward the end of the semester. § See memo B-2-4

Semester III: Data analysis and reporting

Much of the analysis for your reports will be based on straightforward tabulations of frequencies: how many instructors agree or strongly agree with a certain statement, how many students report using a particular kind of instructional device, and so forth. For examples of typical kinds of data analysis and graphical presentations that might be used, refer to Chapters One through Three of Volume One of this report.

The kind of reporting that you do will depend of course on the goals that you had for this project. Sample reports from each of the three sites that took part in the pilot study are provided in Appendix ___.

If your report is to be truly useful, be sure to share your findings with such groups as: the instructional staff, central administration, various ‘partner’ disciplines (the natural sciences, social sciences, engineering, education, etc.) The final report should be the topic of at least one departmental meeting in which possible implications are considered—questions such as , ‘What have we learned ?’, and ‘ What are recommended next steps ?’
Appendices

Appendix A: Further department study options:

*Long-term data-gathering*: To be sure, some very important things can’t be determined in a three or four semester study, but require a long term or ongoing follow up study. Questions for such a follow up study might include:

- What further mathematics courses do students who were surveyed take in subsequent semesters?
- How well do they do in these courses?
- How well do they feel their earlier mathematics courses prepared them for these later courses?
- How well prepared do their instructors feel that these students are?

Similar questions can be studied with respect to partner disciplines, like physics, chemistry, biology, business, etc.

*Sub investigations*:

A project such as this one lends itself to a variety of ‘sub investigations’. Following are suggested optional activities that will supplement the core data collection outlined above.

*What is being taught ?*

To begin to answer this question, it is useful to develop a picture of what your instructional staff (both individually and collectively) think is important for their students to learn in each of the courses. To this end, the following kinds of data can be useful:

- the homework problems assigned by instructors;
- the problems appearing on quizzes, hour exams, midterm exams;
- the problems appearing on the final exam; and
- larger-scale “projects” assigned for students to do, either individually or in groups.

At this point, it is useful to recall that *sampling*, properly used, is a very powerful tool for avoiding information overload. For example, in the case of assigned homework and/or quiz problems, you almost certainly do not want to collect *every* homework problem assigned by *every* instructor for *every* class period. Just how you decide to sample will, of course, depend on your purposes. In the case of homework assignments, you could sample in several different ways: e.g., you might look at homework problems assigned on given, randomly chosen, days; or you might pick a subset of the instructors in a given course to submit their homework assignments. In any case, it’s important to consult with a statistician to help make sure that your samples aren’t *biased*, and are of an appropriate size to give you reliable information.

By itself, just collecting problem sets and tests won’t tell you much. In order to make sense out of this kind of data, you will need to classify the problems according to some kind of *taxonomy*. For already developed taxonomies, see, for example:

**Second International Mathematics Study (SIMS)**

The Second International Mathematics Study taxonomy sorts problems into four classes: recall/recognition; understanding; application; analysis (problem solving). References:


**Third International Mathematics and Science Study (TIMSS)**

- The Third International Mathematics and Science Study has developed a detailed scheme for analyzing textbooks and syllabi. For details, see TIMSS reports at: [http://nces.ed.gov/TIMSS/](http://nces.ed.gov/TIMSS/)
National Assessment of Educational Progress (NAEP)

This framework is used to construct achievement tests for national surveys in mathematics and other fields.

Assessment of student outcomes in mathematics: college level

A rich source of ideas about organizing data about teaching and learning mathematics, written especially for instructors at the college level, is:

Calculus Taxonomy: University of Illinois at Urbana-Champaign

An elaborate taxonomy has been developed and used by the UIUC Mathematics Department Working Group in the Indicators Project for taking a detailed look at the teaching and learning of calculus. The taxonomy provides a way to classify the exercises that occur as homework or as examination problems. It describes each problem in terms of its mathematical content, the cognitive demand that it places on the student and the modes of representation (tables, symbols, graphs) that are used and looks at such additional factors as: the “modes” used to present the problem, (verbal/graphical/symbolic); the modes of representation needed to solve the problem; the types of functions involved (e.g., linear; polynomial; trigonometric; one vs. several variables, etc.). See Appendix C.

Applying an appropriate taxonomy to the problems can help you get a picture of how various courses and/or instructors are balancing such factors as: memory and rote skills (e.g., of formulas and algorithms); routine applications (e.g., just applying a formula to a problem); non-routine applications; proofs; problem solving skills; ingenuity and invention; writing skills; etc.

Examples of important questions to be asked in this connection are:

- What is the variation in cognitive demand across the various different courses offered by the department?
- What is the variation in cognitive demand across different instructors of the same course?
- How does the cognitive demand on the final exam reflect that of the homework and quiz problems in a given class?

For example, consider the last bullet above: Wouldn’t it be food for thought if an instructor assigned overwhelmingly routine, recall/recognition problems for homework, but then asked questions on the final examination that require understanding, or even, perhaps, discovery? Similarly, if the converse were to be true?

Again, for the questions above you may find it instructive to compare: faculty perceptions (via the faculty questionnaire); student perceptions (via student questionnaires); and, finally, a reality check (via class visitations and/or video tapes of selected class sessions.)

§ See Memorandum B*-1 (Calculus Taxonomy administration)
There are also questions relating to outside of class issues and support, like:

- **homework**: How much homework is assigned? How many hours do students devote to their homework assignments? (again a good place to compare student and faculty perceptions!) How useful do students think the homework is in helping them learn the course material?
- **office hours**: how many office hours per week do instructors offer; how much do students avail themselves of these office hours; do students report finding them helpful? (Here again, it might be useful to compare student and faculty perceptions!)
- **tutoring facilities and computer labs**: What kinds of support do they provide? How much are they used? Do students perceive these as valuable
- **The text**: How do students and instructors see as the role of the text? (e.g., as something students are expected to read and understand; primarily as a source of homework problems; etc.).

**Focus/Discussion groups:**

Answers to questions such as the above may be obtained by using focus or discussion groups. Such groups provide yet another way of exploring in depth staff (and student) opinions, views, and feelings about important departmental and programmatic issues. In the context of a variety of opinions and ideas brought to the table in a group discussion, individual members of the group will often express views about things that might never even have occurred to them in an individual interview. In general, people will probably feel more comfortable if you conduct segregated groups for students and faculty. But, for some purposes, you might want to try mixed faculty and student groups. You may also want to include instructional staff from partner departments in some of your focus groups, or, possibly, arrange for a focus group consisting primarily of members of different partner departments.

§ See Memorandum B*-3 (Instructor focus groups)

**Instructor logbooks**: These can be very helpful in developing a picture of what’s happening in a given course, especially in the classroom—at least from the instructor’s point of view. The idea is to ask instructors to keep diary-type records for all, or part, of a course they are teaching. By way of example, you might ask instructors to record their thoughts and impressions of selected classes: How “well” did a given class period go? What kinds of questions did students ask? How was the class time distributed between lecture, discussion, small group work, etc.? How was technology used? etc.

Log books can also be useful for monitoring what’s happening in tutoring and computer labs, e.g., by recording the questions that students ask when they come in for help.

**Classroom visits**: (This can also include video taping of classroom for later observation and analysis)

Direct observations of classrooms is really the only way to get reasonably objective information about classroom processes. In fact, direct classroom observations is a useful way of validating instructor and student perceptions of what goes on in the classroom, as revealed, e.g., in instructor log books, and faculty and student opinion surveys and interviews. Direct classroom observation can help answer questions like: How does the class period “unfold”? Does the instructor seem to be well prepared? How much does the instructor lecture? Are students encouraged to ask questions? Do students seem to be “engaged” in the class? Do students work in cooperative learning groups? If so, do the groups seem to be focused on the work at hand? Are the groups making progress on their assigned tasks? Is technology being used, and, if so, how?

Since classroom processes are generally quite complex, it isn’t all that easy to “observe” them in a way that will be useful for further analysis. In view of this, it is best to use trained, experienced observers. Whether you end up using experienced or inexperienced observers, the following two suggestions may be helpful: (1) Arrange a preliminary session with the observers to discuss with them what aspects of the classroom processes interest you the most (so that observers can focus their observations on those aspects); and (2) have some practice observation runs to work out any glitches that may arise.
It goes without saying that instructors should be notified well in advance of any classroom visitations, and that these visitations should be made as non-intrusive and non-threatening as possible.

For a sample observation form to use in guiding classroom visits, see Appendix D.

**One-on-one interviews:** Individual interviews allow you to prove the view of instructors (and/or students) in much greater depth than is possible with survey instruments (like questionnaires).

Note: Interviews are best conducted by following a prescribed protocol (with some freedom to depart from the protocol if issues arise that deserve further exploration). Interviewees may feel freer to speak their minds if the interviews are conducted by an independent person, rather than a member of the department. For example, you should certainly avoid having students interviewed by a faculty member who may be in a position (now or in the future) to give them grades.

It’s vital to keep good records of all interviews. Best is probably to have someone take written notes, backed up by an audio tape recording. It is also advisable to transcribe the notes and audiotape as soon as possible, while the substance of the interview is still fresh in people’s minds.

**What is learned?**

Surely, ‘What are students learning?’ has to be one of the most basic issues for any department to address. To begin with, one certainly hopes that the question raised here, “What is learned?” and the earlier question, “What is taught?” have closely related and highly correlated answers. Still, even in the best scenarios, the two questions are unlikely to have exactly the same answer. In fact, noting and analyzing the differences between the intent of the teaching, and what students are actually learning, can serve as a sound basis for rational reforms of course syllabi and instructional practices.

To begin with, in connection with programs and courses, you will want to look at grade distributions (and drop-out rates). You can sort data on grades in many different ways, for example for course, by instructor, by student major, by part-time vs full time students, etc.

However, grades, all by themselves, generally won’t tell very much of the “substantive” part of the story—they rarely give you a very precise picture of, or much insight into, what students really know. For example, consider two students, both of whom get a grade of C in a Calculus II course. The first student has an excellent mastery of all of the topics in the course except that he is woefully ignorant about series; the second student, on the other hand, has a so-so knowledge of all the topics in the course.

In order to fill in the picture of “What is learned?” you need to go beyond grades, by looking at actual student work. The first place to look is probably at student answers on final examinations. Again, you can sort such data in a variety of ways. For example, you can do an item analysis for questions that address specific mathematical topics and/or skills that you think are especially important. In fact, the CC group in the Indicators study found it helpful to include several common items on all the final examinations in their multi-section courses in order to be able to compare achievement across sections more meaningfully.

Other student work you may want to look at includes:
- projects or term papers (in courses where these are assigned);
- homework ; and
- answers on quizzes, hour exams, and midterms.

The latter two bullets can give you some insight how students are learning the material as the semester progresses.

**General student attitudes and expectations:** e.g., What beliefs, attitudes, and feelings about mathematics do students bring with them? Do these beliefs, attitudes and feelings change as a function of taking the department’s mathematics courses? What are students’ stated goals in taking mathematics courses? How satisfied are they in general with the department’s offerings? What changes would they like to see?
Student evaluation of specific mathematics courses: e.g., How satisfied are students with the mathematics courses they are currently taking? How difficult do students think these courses are? How much work do they put into their various courses? How comfortable are students with using technology? Do they feel technology enhances their learning of mathematics? How valuable do students find such course components as the lectures, the textbook, doing the homework, working with peers? How many hours do students report spending outside of class for a given course? How valuable are the department’s support services (e.g., instructor office hours; tutoring labs; help via e-mail; etc.)? (For some of these questions, you might want to compare what the students report vs. what faculty think about the same questions!)

In the discussion of “What is taught?” we noted the importance of organizing the homework and examinations questions according to some kind of taxonomy. The same applies to carrying out analyses of what students are learning. A good taxonomy will help give insight into the kinds of mathematical knowledge and skills that students have (or have not) acquired. (Again, see for example the University of Illinois Calculus Taxonomy, Appendix C). Applying an appropriate taxonomy to student responses can help you begin to answer interesting and important questions like the following:

- How well do students perform at various levels of cognitive demand, e.g., how does their performance differ on routine recall questions, straightforward applications, complex problem solving, proof, etc.?
- Does the ability of students to deal with higher cognitive demand questions seem to improve as they move along in their mathematical studies?
- Do varying classroom practices differentially affect student performance on certain types of questions?
- How does the use of technology in teaching seem to affect student performance?

Again, don’t forget that you can use appropriate sampling to drastically reduce the quantity of data you need to collect and analyze.

At this point, it may be good to recall that student work such as homework, tests, even projects, don’t tell the whole story of what students are learning in their mathematics courses. Beyond mastering specific mathematical topics and techniques, most mathematics programs aim at helping students develop rather general “mathematical habits of mind”—ways of approaching and thinking about mathematical problems. That said, it is generally difficult to test for such “higher order competencies” within the time constraints of typical examinations. Therefore, you may want to devise special instruments for getting at some of this. For more on this, see the discussion below (‘How do students use what they have learned?’).

How do students use what they have learned?

This question focuses on the students, and includes such matters as: student demographics; students’ academic backgrounds; their attitudes to mathematics in general; their feelings and opinions about the mathematics courses they are taking; etc.)

But, there are many other questions that can be asked: For example, a question of particular interest to a community college might be: Is there a correlation between the number of hours per week students work outside of school, the number of credit hours they are taking in any given semester, and their mathematics GPA, or drop-out rate?

Student academic backgrounds: e.g., what high schools do students come from? How much, and what kind of, prior mathematics have they taken, and how long ago? With or without technology?

Mathematical Habits of Mind: At the end of the discussion of “What is learned?” we noted that, as a result of their mathematical studies, one hopes that students don’t just end up learning specific mathematical topics and techniques, but also develop certain generalized “mathematical habits of mind” in approaching problems. (This is also sometimes called: “developing mathematical maturity.”) This suggests an important question, but one that seems to be rarely asked--probably because it is so difficult to answer:
As a result of their mathematical studies, how do students go about solving non-routine problems?

Or, perhaps, to put it another way:

How do students actually use the mathematics they have supposedly learned?

Questions of this kind can be especially pertinent in cases where a department is trying to decide whether or not to institutionalize a new, experimental version of a course, and make it a part of the department’s regular course offerings. Suppose, for example, a department is experimenting with various versions of calculus: a traditional lecture-discussion course; a course based heavily on technology (specifically, *Mathematica*); and a third course emphasizing small group work. An important question to ask, therefore, is: In the end, does any of this make a difference, and, if so, how? To help answer this question it would be important to take a detailed look at both: what is being taught (and how it is being taught) and what students have learned. One app they devised some experiments to see whether students coming out of these three versions of calculus—when left to their own devices—approached and thought about calculus problems differently. [For further details, see below, *The “Playpen.”*]

*The “Playpen”:* Above (“How do students use what they have learned?”), we noted the importance of looking at the kinds of generalized “mathematical habits of mind” that students develop and employ in approaching non-routine mathematical problems. It is generally difficult to get much insight into this kind of “knowledge” in the restricted and artificial context of homework problems and course examinations. To see how students actually use the mathematics they have supposedly learned, you may want to replicate the kind of study described below:

At the time of the Indicators Study, the Research University mathematics department was offering three significantly different versions of calculus going: (1) a traditional lecture-discussion course; (2); an experimental, heavily technology-based course; and (3) a course based heavily on small, cooperative learning groups. The department wanted some “hard” evidence on what difference (if any) these various ways of teaching calculus made in terms of the students’ subsequent mathematical behavior. In order to get a handle on this question, the following experiment (whimsically call the *Playpen*) was designed:

First, a small group of student volunteers was chosen—circa two alumni from each of the three versions of calculus from the previous semester. This group was brought together for a preliminary meeting, in which the investigators explained the idea of the study. The students were told that at the next meeting (scheduled for about a week hence), they would be presented (in writing) with three or four somewhat non-routine problems to solve. The room in which they would be working would be equipped with a variety of aids, including: textbooks, reference materials, graphing calculators, and computers with assorted software (e.g., *Mathematica*, which at least some of them were familiar with). They could use any or all of these tools in any way they thought might be useful. They were also free to work alone, with partners, or in groups of their own choosing.

The actual *Playpen* session lasted about 1½ hours. The proceedings of the *Playpen* session were videotaped for later viewing and analysis. In addition, several observers were stationed around the room to take notes on how the students went about solving the problems they had been given: Who did they chose to talk to? What did they talk about with each other? Which of the tools in the room did they use? etc. Before the end of the session, students were asked to write up their solutions, which were collected by staff for later analysis.

A week after the *Playpen* session, the students were again brought back together for a debriefing meeting. The investigators solicited the students’ views and feelings about their *Playpen* experience, as well as discussing the mathematics involved in the problems they had been given.
**Focus/Discussion groups:** Such groups provide yet another way of exploring in depth student and faculty opinions, views, and feelings about important departmental and programmatic issues. In the context of a variety of opinions and ideas brought to the table in a group discussion, individual members of the group will often express views about things that might never even have occurred to them in an individual interview. In general, people will probably feel more comfortable if you conduct segregated groups for students and faculty. But, for some purposes, you might want to try mixed faculty and student groups. You may also want to include faculty from partner departments in some of your focus groups, or, possibly, arrange for a focus group consisting primarily of members of different partner departments.

Note: For a freer discussion, consider using an independent leader. Again, it is very important to keep records of what transpires in your focus/discussion groups. Producing intelligible, usable records for groups is more difficult than for individual interviews. Two human recorders, and a videotape recording can help, e.g., in keeping straight who said what.

Examples of questions to raise with focus groups could include:

- **partner discipline department questionnaires**, that solicit information on the mathematical needs of partner discipline departments, as well as their degree of satisfaction with the department’s current programs and service courses.

Surveys are limited in terms of the kind and quality of information that you will obtain. To get more in-depth (and probably more useful) data, you might want to follow up the surveys with interviews, classroom observations or focus groups.

In preparing to do interviews or focus groups, you should prepare a list of key points that you intend to cover. Consider doing a ‘dry run’ with a few individuals in order to refine the process and help ensure that you get the kind of information that you need.

You should also develop a calendar for your class visitations. Instructors should be notified well in advance of the date when their classes will be observed.

-- Notify the prospective **interviewees**, including dates, and what the interviews will be about.

-- If you have planned on **focus groups**, start selecting the members of the groups around mid-term, and schedule them for the last month of the semester—but well before final exams. You may want to have several different kinds of focus groups, e.g., for students, for departmental faculty, for partner discipline faculty, and perhaps one mixed faculty/student group.

§ See memo … B*-3 Student focus groups

**Tests, homework, portfolios, projects, etc:** Collecting homework assignments, student projects, quizzes, tests, and the final exam can give you important insight into a given course from two points of view:

(a) **instructor expectations** (i.e. what are instructors expecting the students to learn?); and

(b) **student achievement** (i.e., what are the students actually learning?).

While these kinds of data are extremely important for getting insight into what’s going on in your courses, it’s also important not to overwhelm the instructors with too many onerous data-gathering chores, nor, for that matter, to get overwhelmed yourself with the data you have collected. Some suggestions on how to use judicious **sampling** follow:

- For **homework and quizzes**, it’s unlikely that you will want to see much of the actual student work—just having the instructor’s assignments will generally suffice. (You may be interested in class averages
on a few of the homework assignments and/or quizzes; and you might want to sample actual student work by collecting and analyzing just one or two homework assignments and/or quizzes during the course of a term.)

- Even for the hour exams, it will usually be the exams (rather than the students’ actual answers) that you will want to collect and analyze. However, you may be interested in seeing the overall grade distributions for each exam and each class and instructor. (see also: 8. Grade distributions [below]).

- For the final examinations, you may well want to actually collect and analyze all, or at least, some of the student responses. For courses that have multiple parallel sections, it is useful to get some insight into variations across various sections and instructors. That’s easiest to do if all the sections give a common final. If common exams are not the tradition in your department, you should at least try to persuade the various instructors to include some common questions (the CC people used six) in their particular version of the final exam.

- For classes that involve projects of one sort or another, you should keep an inventory of the types of projects assigned, as well as at least samples of student work. Information that is useful to gather for projects includes: individual vs. group projects; the length of time required to complete the project; the mathematics involved; the technology tools used in the project; etc.

Recall, once again, that for these kinds of data to be really useful, you will need to sort the problems according to some kind of taxonomy. Depending on your particular department’s interests and needs, the taxonomy can be more or less elaborate, but, at a minimum, it should tell you something about the “intellectual demand” of the various problems, e.g., memory/recall, understanding, application, transfer, discovery. [see also above, “What is taught?”]

Long term Effects
The most important effects of a mathematics course or a mathematics program don’t play out in the short term, but only over the passage of time. Thus, it is very helpful (even though it may be expensive) to continue the study into subsequent semesters (and even years), and to follow the subsequent “careers” of the alumni of the study as long as one can. Questions of interest in such a follow-up study include:

- Do alumni continue their mathematical studies—either by taking more advanced mathematics courses proper, or courses in mathematics-rich disciplines?
- Do they feel that their prior mathematics courses have prepared them well for these later courses?
- Looking back at their earlier mathematics course(s), how do they feel about them now?

Post-study follow-ups:

By definition, long-term follow-ups require a lot of time, and you may be not be able or willing to spare that kind of time for your study. But, at the very least, you should do some follow-up in the semester immediately following the main portion of the study. First, you can use the school’s central data base to determine which of the alumni of your study have gone on to take further mathematics or mathematics-intensive courses. You can then use follow-up survey questionnaires to determine how these alumni—from their current vantage point—feel generally about the mathematics courses they took during the study, and, in particular, how well they feel these courses prepared them for their current mathematical undertakings. You may also want to consider scheduling some follow-up individual interviews and/or focus groups with alumni of your study to deal with these kinds of issues.

In addition to surveying those alumni who have “gone on” to further mathematics courses, you may also want to follow the example of the CC group in the Indicators Project, and survey students who did poorly during the semester (or year) of the study (got D’s or F’s) or students who withdrew. Doing this may help you to develop useful profiles of at-risk students and then develop some ways of helping these students succeed.
Appendix B. Selected memoranda for Indicators Project Management

Memorandum B-1-1 (‘Setting the stage for the Indicators Project’)

Department of Mathematics

12 May ____

To: Selected Staff
From: Indicators Project Director
Re: The Indicators Project

We are one of three mathematics departments participating in an NSF funded project aimed at developing ways to evaluate undergraduate mathematics programs. Our focus will be a comparative study of our calculus sequences. We are one of three campuses around the country that is taking part in this project. I am in charge of our portion. This Summer will be spent formulating our methodology and we hope to begin data collection this Fall.

I am writing to solicit suggestions from you about how we should be going about this task. For example, what criteria give insights into student success in mastering the calculus, both near and long term? All of us have favorite problems that we think help get to the essence of this or that topic. It would be very helpful if you could share some of those with us.

Please feel free to send me suggestions about any aspect of this project that you wish to comment on.

---

We wish to Professor Paul Weichsel, Associate Chair, Department of Mathematics, University of Illinois at Urbana-Champaign,
Memorandum B-1-2  (pilot study—trying out data collection procedures)

Department of Mathematics

19 June ___

To: Selected Staff  
From: Indicators Project Director  
Re: Calculus Study Project

As part of a National Science Foundation-funded undergraduate mathematics education study, the Department of Mathematics will be closely examining all of the calculus offerings at the University of Illinois. I need a little bit of your time on behalf of the goals of this project.

In the fall semester, we will be collecting information about calculus students and calculus classes. The information we collect will be used to measure and evaluate the students' attitudes and knowledge. This summer we will be testing out our data collection techniques. We would like to ask you for two favors.

**Student Surveys:** We would like you to pass out two 10-minute surveys to your students, once at the beginning of the summer and once at the end of the summer. Copies of the first survey are in the envelope attached to this letter. Please pass out the surveys in class on Friday, 20 June or Monday, 23 June, and return the completed surveys to my mailbox no later than Tuesday, 24 June. The survey should take the students no more than 10 minutes to complete. Please note that each questionnaire has a student's name on the cover page and it is important that each student complete his/her questionnaire. Some blank copies are enclosed for those students not on your roster.

**Instructor Meeting:** Please come to a meeting for the summer calculus teachers in which we will be describing more about the Indicators Project and some of the other information we will need from you this summer. The meeting will be: Thursday, June 26 at 1 p.m. in ______. If you cannot attend, please let me know.
Memorandum B-2-1 (administration of first student survey: beginning of semester)

Department of Mathematics

26 August _____

To: All instructors of lower division courses
From: Indicators Project Director
Re: The Indicators Project

Attached is an envelope of questionnaires that I am requesting you distribute to your students by Wednesday, 28 August at the latest. They should take no more than 10 minutes for the students to complete and should be collected the same day and returned to _________ in the main office.

This is the first activity of the Indicators Project this semester. Most of you know that the purpose of the Project is to evaluate the effectiveness of our calculus instruction in all of its forms. I am the director of the project for our department and I am hopeful that we will gain valuable information about our calculus instruction as well as develop a methodology for such assessments that could be used by other institutions around the country.

This survey is the first of two student surveys that you will be asked to administer this semester. The other will be at the end of the semester. The purpose of these surveys is to tell us something about the background and attitudes of our calculus students and to assess possible changes in attitudes during the semester.

Please note that each questionnaire contains a statement about the voluntary nature of this project. I would like you to read the statement aloud to your class as you pass out the forms.

I very much appreciate your cooperation in this matter. If you have any questions or suggestions about the project, please communicate with me in any way you like and I will be happy to share information and my thoughts with you.
Memorandum B-2-2: (cover page for student survey: beginning of semester)

Department of Mathematics

Indicators Study: Calculus Evaluation

September ___

Student Survey I

Name:

Student ID #:

This year, the Department of Mathematics is conducting a study about the different types of calculus instruction on our campus. This study will not be successful without your help.

Please take a few minutes to complete the attached survey.

YOUR INSTRUCTOR WILL NOT SEE THE RESULTS OF THIS SURVEY

Thank you for your cooperation.
Memorandum B-2-3  (administration of end of semester student survey)

Department of Mathematics

24 November

To: Lower division mathematics instructors
From: Indicators Project Director
Re: The Indicators Survey (for students)  II

Attached is an envelope of questionnaires that you should distribute to your students during the last week of class. This is the follow-up to the one that you distributed at the beginning of this semester. The survey should take no more than 10 minutes to complete. Please be sure to have the students complete it during class, and ask a class member to collect all forms and return them to _______ in the Main Department Office (Instructors should not see the completed student forms).

Each survey has a statement on the first page that you need to read to your class. It deals with the issue of voluntary participation.

I very much appreciate your cooperation in this matter. If you have any questions or suggestions about the Indicators Project please communicate with me in any way you like and I will be happy to share information and my thoughts with you.
Memorandum B-2-4 (administration of instructor survey)

Department of Mathematics

1 December

To: Instructors of lower division courses
From: Indicators Project Director
Re: The Indicators Survey

We are now reaching the end of data gathering for the Indicators Project. Attached is an instructor survey which I am requesting that you fill out at your earliest convenience. It should take about 20 minutes to complete.

Please return the completed survey to ___________ in the Main Department office by 4 December at the latest. Please note that questions about your classroom activities refer to the course that you are teaching this semester.

A very similar survey is being used at the other two sites that are involved in the Indicators Project. As with every other aspect of this project, this activity is voluntary. We need the information in order to factor in possible effects of our instructor profile compared to those at the other two institutions.

Thank you very much for your cooperation. Please know that I very much appreciate your cooperation in this matter. And, as with other aspects of this project, please communicate with me in any way you like and I will be happy to share information and my thoughts with you.
Memorandum B*-1 (Calculus Taxonomy)

To: All instructors of lower division courses
From: Indicators Project Director
Re: The Indicators Project: Hour Exams

As my next request to you for help in the Indicators project is perhaps the most demanding I will make, let me preface it by providing some background. As you know we are trying to assess the effectiveness of our various calculus programs….The survey that you administered provides a baseline description of the students we are studying. Together with admissions data that we are obtaining from the campus we will have a good insight into the educational background, the expectations and the attitudes about their mathematics program that the calculus students bring with them to your classes.

The homework data that we are collecting during the semester will tell us what we, as a department, think are the problems that students ought to try to solve en route to a successful calculus experience. In the same vein we want to know what problems appear on our hour exams and how we evaluate our students' performance. Our intention is to classify all of the exercises that appear on homework assignments and on exams into a number of categories, a *CALCULUS TAXONOMY* as it were, and thereby gain a rich description of our program. I want to emphasize that in all of this WE HAVE NO INTEREST WHATSOEVER IN EVALUATING ANY INSTRUCTOR, WHETHER TEACHING ASSISTANT OR PROFESSOR. We are only interested in an aggregate description of calculus as offered in our department and I give you my word that no information about any individual will ever be transmitted beyond the confines of my office. I would therefore like to request that you furnish us with the following information.

1. A copy of each of your hour exams.
2. Each student's score on each of the problems on each exam

A simple way to accomplish this is to prepare your exams so that page 1 has the student's name and social security number and the number of points that you awarded that student for each of the problems on the exam. (I am sure that some of you already do this as a matter of course.) You could then copy those front pages after you are done grading and send them to me. I am including the front page of one of my old hour exams as an example.

My final request for information will be that you send me all of your final exam papers when you are done grading at the end of the semester. I will retain them in my office and they will be available to you for a year following the end of the semester as required by the campus.
Memorandum B*-2 (Focus group for instructors)

Department of Mathematics

9 April

To: Instructors of lower division courses
From: Indicators Project Director
Re: The Indicators Project

As you all know, we have been collecting a great deal of "hard" data on our calculus courses for almost a whole academic year as part of the Indicators Project. We would now like to utilize a data instrument that has been used successfully in variety of experiments to extract the sort of information that is not always available through surveys and counting. I'm referring to the "focus group".

We would therefore like to invite you to attend an hour-long guided conversation with a group of faculty and TAs who have had extensive experience teaching calculus in more than one format.

We invite you to meet on Thursday, 16 April, from 4 to 5 PM in Room ___. Can you come? I would appreciate your response, one way or the other, as soon as possible.

Refreshments will be served.

Here are some questions to help focus your thoughts before the meeting:

1. What do you think your students learned or failed to learn from the various versions of calculus that you have taught?
2. How would you advise students who were choosing from among the various versions of our calculus courses?
3. How did you change your teaching style to accommodate the different versions of calculus that you taught?
4. What advice would you give to colleagues who were considering teaching calculus in a nontraditional format?
5. Have there been any long term effects on your teaching from your experiences with either of the ‘reformed calculus’ courses?
6. Do you have an opinion about whether the Department should continue to offer calculus in several different formats?
Memorandum B*-3 (Invitation to students—focus group)

Department of Mathematics

21 October

To: Selected calculus students
From: Indicators Project Director
Re: The Indicators Project

The Department of Mathematics is engaged in an extensive National Science Foundation-funded project to assess our calculus program. An important aim of this project is to determine how students learn in the different versions of our courses. You have been selected from a group of students who successfully completed Calculus II last Spring to participate directly in this project. If you agree, you will be asked to join a small group of your peers (between 8 and 12) and spend an hour in a directed conversation, usually called a "focus group", dealing with your experiences in your math courses.

In addition to giving the department valuable insights into our calculus program, you will have the opportunity to express yourself in a friendly and non-threatening atmosphere. As a token of our appreciation for your participation you will be paid $10.

We have scheduled 4 of these sessions during the next two weeks; their dates and times are:

27 October  4-5 PM
29 October  5-6 PM
3 November  4-5 PM
5 November  5-6 PM

Please make a first and second choice from among those dates in your response.

In order for us to produce accurate records of these sessions, we will record each one. Let me assure you that we have no interest in identifying individuals and the reports of these sessions will contain no information that can be linked to any participant.

Finally, let me urge you to respond to this message whether you intend to participate or not.

Thank you
Appendix D: Classroom observation form (courtesy of T. J. Murphy, Department of Mathematics, University of Oklahoma)

<table>
<thead>
<tr>
<th>Instructor: __________________________</th>
<th>Observer: __________________________</th>
</tr>
</thead>
<tbody>
<tr>
<td>Course Number: ____________</td>
<td>Observation Date: ________________</td>
</tr>
<tr>
<td>Course Name: ________________</td>
<td>Observation Time: ________________</td>
</tr>
</tbody>
</table>

**Physical Setting**

- no./type of student seats available: ______  no. of students attending/enrolled: ______  
  (fixed, moveable, tables, tiered rows, computer station)
- resources available to instructor: __________________________
  (chalkboard, overhead projector, calculator, computer (desktop or laptop), projection system)
- resources available to students: __________________________
  (paper, textbook, chalkboard, calculator, computer (desktop or laptop), projection system)

**Instructor Primary Activity(s)**

- The instructor intended to: __________________________
  (lecture, interactively lecture, circulate to answer questions, observe students)
- The instructor did: __________________________
  (lecture, interactively lecture, circulate to answer questions, observe students)
- The instructor also (“misbehaviors”): __________________________
  (was late or unprepared, mumbled, interacted poorly with students, interrupted student momentum)

**Student Primary Activity(s)**

- Students were to be: __________________________
  (taking notes, interacting with instructor, interacting with each other)
- Students were: __________________________
  (taking notes, interacting with instructor, interacting with each other)
- Students also (“misbehaviors”):
  (left early, slept, played with e-mail or games, engaged in off-task conversations)
- no. of students participating: ____  no. of students misbehaving and how often: ____  
  (less than half, half, more than half)  (once, occasionally, frequently, excessively)

**Primary Media Used**

- by instructor: __________________________
  (chalkboard, overhead projector, calculator, computer (desktop or laptop), projection system)
- by students: __________________________
  (paper, textbook, chalkboard, calculator, computer (desktop or laptop), projection system)
The Calculus Taxonomy

by

Paul M. Weichsel

University of Illinois at Urbana-Champaign

1409 West Green Street

Urbana, IL 61801
1. Introduction

Is there a way to describe the "essence" of a mathematical problem that might appear as an exercise in a textbook, electronic or conventional, a homework problem formulated by the instructor or on an exam in a mathematics course? The answer to such a question depends on the anticipated purpose of such a description. The scheme that we describe below, emerged as part of an NSF-sponsored study, called the "Indicators Project", to develop a methodology for the analysis and assessment of the first two years of undergraduate mathematics. We at the University of Illinois in Urbana-Champaign axe focusing on our "engineering calculus" program. We are now in the last stages of an extensive data collection activity which includes, among other things, all homework assignments, hour exams and final exams for all sections of the relevant courses. The size of the cohort that we are studying is about 4500 students. A major tool in our analysis plan is a "Calculus Taxonomy" which associates with each calculus exercise a 12-tuple of integers. We intend to use this classification to describe certain attributes of the "delivered curriculum" in our calculus classes, to obtain some measure of variability among different sections of the same course, and to analyze and compare the textbooks that we use in different versions of the same course.

2. The Taxonomy

To each exercise we associate a partitioned vector \( \tau \) in \( \mathbb{R}^{12} \):

\[
\tau = (C : D : R)
\]

and we refer to the vector \( r \) as the encoded problem.

Each entry in \( r \) has the following meaning:

- **C:** the vector \( C \) is a 4-tuple which describes the "content" of the exercise in terms of the text from which it was taken, if that is relevant, the number of the section of a virtual textbook in which it would most probably appear, the number of variables involved, and the function that appears in the problem.

- **D:** The vector \( D \) is a 4-tuple which represents the "cognitive demand" of the exercise. Its components describe whether the problem is placed in a well defined context, such as a section of a textbook, the number of significant
4. Cognitive Demand

The vector $D = (d_1, d_2, d_3, d_4)$ is the most problematic of $C, D,$ and $R$. After giving a brief description of its components we will illustrate the encoding process by a series of examples.

$d_1$, denotes the extent to which the problem appears in a context that suggests the nature of the solution.

- $d_1 = 0$ when the problem appears on an exam with no hint given
- $d_1 = 1$ when the section of the text in which the problem appears is known to the student or when the problem appears on a exam with a hint
- $d_1 = 2$ when the context is strongly indicated as in a problem with multiple parts, each being a step in the solution

$d_2$ denotes the number of different steps required in the solution of the problem. If each step is identical to the one before, such as a multiple differentiation, we count that as one step.

$d_3$ is 1 or 0 according as the problem is a significant application involving a meaningful (to the student) real world situation and requiring some nontrivial modeling or not. Thus a problem like "Engineers have determined that the cost of building a tower n feet tall is given by the equation $A = B$, find . . ." is not considered an application and $d_3$ is given the value 0. Similarly, "Mary is 3 years older than Tom whose age is 3/2 that of Bill, . . ." is not an application and $d_3 = 0$ for that problem. A good example of a problem for which $d_3 = 1$ is the following,

The rate at which the world's oil is being consumed is increasing. Suppose that the rate (measured in billions of barrels per year) is given by the function $r(t)$, where $t$ is measured in years and $t = 0$ is 1 January 1990. Write a definite integral that represents the total quantity of oil between the start of 1990 and the start of 1995.

$d_4$ measures the level of difficulty with 1 denoting simple recall and/or the applica-
tion of elementary algorithms to 4 representing a problem reasonably situated in a calculus course, but whose solution requires a "leap of imagination" on the part of the student. Here are some examples.

\[ d_4 = 2 \]

How long will it take for an investment to double in value if the interest rate is 6% compounded continuously?

\[ d_4 = 3 \]

What is the area of the largest triangle that can be formed in the first quadrant by the x-axis, the y-axis, and a tangent to the graph of \( y = e^{-x} \)?

5. Representational Form

In each of the following representational categories, 0, 1, 2 or 3 is given if: 0: that representation is not present either in the problem or solution 1: the student is required to interpret that representation 2: the student is required to construct that representation 3: both 1 and 2 above

r1: **graphical** means a graph, pie chart, scatterplot or similar representation

r2: **numeric** or **tabular** means a set of numerical data organized in a table or in some other form

r3: **symbolic** includes mathematical expressions in symbolic form

r4: **verbal**
1. Introduction

Is there a way to describe the "essence" of a mathematical problem that might appear as an exercise in a textbook (conventional or otherwise), a homework problem, or an exam question? The "essence" of a math problem lies in its input, solution process, and student output. As such, the following taxonomy has been devised in order to successfully describe these three separate components.

The taxonomy scheme described below explains how to go about associating with each problem a series of twelve integers which attempt to completely describe the three components: content, cognitive demand, and representational form. After such an encoding has been completed, it is then possible to describe the differences between individual problems, chapters, and books through analysis of the similarities and differences of their encodings.

Following is a table of expressions used throughout this and related documents, along with their usage.
## 1.1 Glossary

<table>
<thead>
<tr>
<th>Expression</th>
<th>Usage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taxonomy</td>
<td>The process and methodology used to assign with each problem a vector that describe its various aspects.</td>
</tr>
<tr>
<td>Content</td>
<td>The subvector that describes the mathematical content that the problem presents within the question.</td>
</tr>
<tr>
<td>Cognitive Demand</td>
<td>The subvector that describes the mathematical difficulty required in solving the problem.</td>
</tr>
<tr>
<td>Representational Form</td>
<td>The subvector that describes the various mathematical forms found in the body or solution of the problem.</td>
</tr>
<tr>
<td>Encoding</td>
<td>The assignment of a vector to a problem.</td>
</tr>
<tr>
<td>Vector</td>
<td>The set of integers that describe the problem's content, cognitive demand, and representational forms.</td>
</tr>
</tbody>
</table>
1.2 Taxonomy Assumptions

Due to the difficult nature of attempting to assign accurately and meaningfully each problem a vector, several assumptions must be made to prevent the taxonomy's complexity from growing beyond its usefulness. The following assumptions will be made throughout the design and implementation of the taxonomy.

1.2.1 Each Problem Has Only One Solution

Although many problems can be solved using different methods, it is reasonable to assume that the typical student will solve the problem with a particular solution method, given the context of the problem. The encoding of the problem must assume, for the sake of feasibility, that such a solution is the only solution.

Find the centroid of the plane region bounded by the curves $x = 0, x = 4, y = 0,$ and $y = 6$. Assume that the density is $\rho \ldots 1$ for the region. Edwards and Penney, Section 15.5, #1

Given that the problem occurs in the chapter where the student is taught to utilize the formulas

\[
\bar{x} = \frac{1}{m_R} \int x \rho(x,y) dA \quad \text{and} \quad \bar{y} = \frac{1}{m_R} \int y \rho(x,y) dA,
\]

it is reasonable to assume that the typical student will solve the problem using these formulas.

However, it is equally correct to interpret the problem as the center of mass of a rectangle, which is located at the rectangle's center from simple algebraic and geometric demonstration.

Due to the context of the problem, it should be encoded as if the first solution mentioned was the only solution.

1.2.2 Not All Multi-Part Problems are the Same

Although many problems are presented with multiple subparts, not all should be treated in the same manner. Some multiple part problems contain independent subparts, and hence should be encoded as multiple problems. Other multiple part problems contain dependent subparts (that often lead to a conclusion), and hence should be coded as a single problem with a high value for the number of steps required in the solution.
Let \( h(t) \) represent the height, in meters above ground level, of an object (a helium balloon, for instance, or a cannonball, an airplane, or a toy rocket) at time \( t \) seconds. Let \( v(t) = h'(t) \) represent the vertical velocity, in meters per second, at time \( t \) seconds. Note that \( v(t) > 0 \) means the object is rising; when \( v(t) < 0 \), the object falls.

(a) Solve the IVP \( v'(t) = 1; v(0) = 0 \). Assuming these conditions, what is the object's upward velocity at \( t = 10 \) seconds? [HINT: if \( v' \) is a constant function, then \( v \) is a linear function.]

(b) Solve the IVP \( h'(t) = -3; h(0) = 0 \). Find and interpret \( h(10) \). [HINT: If \( h' \) is a constant function, then \( h \) is a linear function.]

(c) Verify by differentiation that for any constant \( C \), \( v(t) = \frac{100}{t + C} \) solves the DE \( v' = -0.01v^2 \).

(d) Suppose that \( v' = -0.01v^2 \) and \( v(0) = 5 \). What's \( v(30) \)? What's \( v(80) \)? Interpret your answers in physical language. [HINT: See the previous part.]

Ostebee Zorn, Section 4.1, #16

The above problem presents a problem where there are both dependent and independent subparts. Because parts (a) and (b) are independent in that their solution is independent of the other subparts, they should be encoded as separate problems. However, parts (c) and (d) are dependent in that their solution is dependent on each other, and hence should be encoded together as a single problem.
1.3 Taxonomy Vector Basics

Each exercise is associated with a series of 12 integers (a vector in $\mathbb{R}^{12}$) that represent various aspects of the problem. The 12 numbers are grouped in three sets of four numbers, each group describing a different aspect of the exercise.

1.3.1 The Content Vector

The subvector $C$ is an integer 4-tuple ($C_1,C_2,C_3,C_4$) whose values describes the mathematical content that the problem presents within the question. It's purpose is to represent the content of the question, and should in no way include information about the solution of the problem.

The 4-tuple's meaning is according to the following table:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>Text from which the problem was taken</td>
</tr>
<tr>
<td>$C_2$</td>
<td>Section of the virtual textbook where the problem would appear</td>
</tr>
<tr>
<td>$C_3$</td>
<td>Number of variables involved in the question</td>
</tr>
<tr>
<td>$C_4$</td>
<td>Type of function involved in the question</td>
</tr>
</tbody>
</table>

1.3.2 The Cognitive Demand Vector

The subvector $D$ is an integer 4-tuple ($D_1,D_2,D_3,D_4$) whose values describes the cognitive demand of the problem. It's purpose is to represent the difficulty of the problem and the factors that contribute to its difficulty.

The 4-tuple's meaning is according to the following table:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1$</td>
<td>Extent to which the problem is in context</td>
</tr>
<tr>
<td>$D_2$</td>
<td>Number of steps required to complete the problem</td>
</tr>
<tr>
<td>$D_3$</td>
<td>Whether the problem modeling of a real-world situation</td>
</tr>
<tr>
<td>$D_4$</td>
<td>Difficulty level required in solving the problem</td>
</tr>
</tbody>
</table>
1.3.3 The Representational Form Vector

The subvector $\mathbf{R}$ is an integer 4-tuple $(R_1, R_2, R_3, R_4)$ whose values describe the representational forms utilized in the problem. Its purpose is to represent the usage of graphical, tabular, symbolical, and verbal expression within the question and the problem’s solution.

The 4-tuple's meaning is according to the following table:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$</td>
<td>Extent to which graphical representation is required</td>
</tr>
<tr>
<td>$R_2$</td>
<td>Extent to which numeric/tabular representation is required</td>
</tr>
<tr>
<td>$R_3$</td>
<td>Extent to which symbolical representation is required</td>
</tr>
<tr>
<td>$R_4$</td>
<td>Extent to which verbal representation is required</td>
</tr>
</tbody>
</table>
2. Vector Definitions

As mentioned in the introduction, each problem is encoded by a 12-tuple of integers. The values that these categories assume are enumerated in detail below.
2.1 $C_1$ - Text

This variable denotes the text where the problem appears.

The value of this variable is assigned according to the following table:

<table>
<thead>
<tr>
<th>Value</th>
<th>Source of the Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Exam or other resource</td>
</tr>
<tr>
<td>1</td>
<td>Ostebee and Zorn</td>
</tr>
<tr>
<td>2</td>
<td>Stewart</td>
</tr>
<tr>
<td>3</td>
<td>Calculus &amp; Mathematica</td>
</tr>
<tr>
<td>4</td>
<td>Sallas and Hille</td>
</tr>
<tr>
<td>5</td>
<td>Edwards and Penney</td>
</tr>
</tbody>
</table>
2.2  \( C_2 \) - Subject Matter

This variable denotes the subject matter of the problem. There will be a virtual textbook, which will be the union of all of the calculus topics taught at the University of Illinois.

The value of this variable will correspond to the topic in the virtual textbook.
2.3  C₃ - Number of Variables

This variable denotes the maximal number of variables that are manipulated within the problem.

The value of this variable is the maximal number of variables that are manipulated within the problem.

**Example #1:**

Differentiate the function \( y = \sqrt{5x} \).

Stewart, Chapter 2.2, #17

The value of \( C_3 \) for this problem would be 1. Although the variable \( y \) appears in the problem, it is a dependent variable and thus should not be counted.

**Example #2:**

Find \( \frac{dy}{dx} \) by implicit differentiation if \( x^2 + xy + y^2 = 9 \).

Edwards and Penney, Chapter 3.8, #6

The value of \( C_3 \) for this problem would be 2. The variables \( x \) and \( y \) are both counted in the problem because the student is required to manipulate both variables when using implicit differentiation.
2.4 $C_4$- Type of Function

This variable denotes what types of functions are given in the body of the problem that the student must use in order to solve the problem. When a problem contains more than one type of function, the problem should be assigned the highest possible numerical value that applies.

The value of $C_4$ should only represent the functions given and should not represent any function produced as part of the solution to the problem.

The value of this variable is assigned according to the following table:

<table>
<thead>
<tr>
<th>Value</th>
<th>Type of Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Constant</td>
</tr>
<tr>
<td>10</td>
<td>Linear</td>
</tr>
<tr>
<td>20</td>
<td>Quadratic</td>
</tr>
<tr>
<td>30</td>
<td>Polynomials of degree 3 or higher</td>
</tr>
<tr>
<td>40</td>
<td>Rational (Ratio of polynomials)</td>
</tr>
<tr>
<td>50</td>
<td>Algebraic</td>
</tr>
<tr>
<td>60</td>
<td>Trigonometric or Inverse trigonometric</td>
</tr>
<tr>
<td>70</td>
<td>Transcendental</td>
</tr>
<tr>
<td>80</td>
<td>Combination including at least one of the 60 or 70</td>
</tr>
<tr>
<td>90</td>
<td>Compound of at least one of 60 or 70</td>
</tr>
<tr>
<td>100</td>
<td>Piecewise-defined, Absolute value, Greatest integer</td>
</tr>
<tr>
<td>110</td>
<td>Abstract</td>
</tr>
<tr>
<td>120</td>
<td>Graphical</td>
</tr>
<tr>
<td>130</td>
<td>Tabular</td>
</tr>
<tr>
<td>140</td>
<td>Series</td>
</tr>
<tr>
<td>150</td>
<td>Miscellaneous (Zeta, Hyperbolic, Recursive)</td>
</tr>
<tr>
<td>160</td>
<td>Differential Equations</td>
</tr>
<tr>
<td>170</td>
<td>Parametric</td>
</tr>
<tr>
<td>990</td>
<td>None given explicitly or implicitly</td>
</tr>
</tbody>
</table>

Example #1:

Evaluate the antiderivative $\frac{3}{x^2+1} \, dx$. Check your answer by differentiation.

Ostebee Zorn, Chapter 6.1, #4

The value of $C_4$ for this problem would be 4, as the function presented is the ratio of polynomials.
Although the solution contains an inverse trigonometric term, we do not count it.
Example #2:

Differentiate the function $y = \tan(e^{3x-2})$

Stewart, Chapter 3.1, #37

The value of $C_4$, for this problem would be 8. The existence of the $\tan$ function would warrant a value of 6, the existence of the $\exp$ function would warrant a value of 7, and the compound of these functions warrants a value of 8. Since the value of the compound function is greatest, we use it.
2.5 $D_1$ - Context

This variable denotes the extent to which the problem appears in context that suggests the nature of the solution.

The value of this variable is assigned according to the following table:

<table>
<thead>
<tr>
<th>Value</th>
<th>Context of the Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Presented completely out of context</td>
</tr>
<tr>
<td>1</td>
<td>Presented with one suggestion of context</td>
</tr>
<tr>
<td>2</td>
<td>Presented with two suggestions of context</td>
</tr>
<tr>
<td>3</td>
<td>Presented with three suggestions of context</td>
</tr>
</tbody>
</table>

Although higher values are not included in the table, arbitrarily large values can be assigned to a problem.

Example #1:

Evaluate the integral $\ln(1 + x)dx$

Edwards and Penney, Chapter 9 Miscellaneous Problems, #23

The value of $D_1$ for this problem would be 0, as the problem is presented completely out of context. Because it appears in the miscellaneous section of a chapter that presents a variety of integration techniques, the student has no suggestion as to how to solve the problem.

Example #2:

Find the radius of convergence of the power series

$$x \sum_{j=1}^{\infty} \frac{x^j}{2^j}$$

Ostebee Zorn, Chapter 11.5, #2

The value of $D_1$ for this problem would be 1. Because it appears in the chapter that introduces power series and utilizing the ratio test to find the radius of convergence, the student has one suggestion as to the solution of the problem.
Example #3:

Prove that if \((c, f(c))\) is a point of inflection on the graph of \(f\) and \(f''\) exists in an open interval that contains \(c\), then \(f''(c) = 0\). [Hint: Apply the First Derivative Test and Fermat's Theorem to the function \(g = f'\).]

Stewart, Chapter 4.4, #37

The value of \(D_I\) for this problem would be 2. Because it appears in the chapter that introduces points of inflection and a hint is provided, the student has two suggestions as to the solution of the problem.
2.6 $D_2$ - Steps

This variable indicates the number of steps required to solve the problem. In determining this value, repetition of an algorithm should be counted only if its repeated use places some significantly new cognitive demand on the student.

The value of this variable is the number of steps required to solve the problem.

Example #1:

Find the first 73 derivatives of
$$f(x) = x - x^2 + x^3 - x^4 + x^5 - x^6.$$  
Stewart, Chapter 2.7, #27

The value of $D_2$ for this problem would be 1. Because each successive derivative requires no extra cognitive demand, only one step should be counted.

Example #2:

Evaluate the integral
$$\int x \cos x \, dx.$$  
Stewart, Chapter 7.1, #2

The value of $D_2$ for this problem would be 2. Although both steps of the solution involve the same algorithm, integration by parts, additional cognitive demand is required in the repeated application of the algorithm.
2.7 $D_3$ Applications

This variable indicates whether the problem is a significant application involving a real-world situation and/or requires some non-trivial modeling. The problems are given the value of 1 if such application is present and the value of 0 elsewhere.

Example #1:

Find $f(g(x))$ and $g(f(x))$ when $f(x)=1-x^2$ and $g(x)=2x+3$.

Edwards and Penney, Chapter 2.4, #1

The value of $D_3$ for this problem would be 0. Clearly no real world situation is being modeled or manipulated.

Example #2:

The frequency of vibrations of a vibrating violin string is given by $f=\frac{1}{2L} \sqrt{\frac{T}{\rho}}$, where $L$ is the length of the string, $T$ is its tension, and $\rho$ is its linear density. Find the rate of change of the frequency with respect to (a) the length (when $T$ and $\rho$ are constant), (b) the tension (when $L$ and $\rho$ are constant), and (c) the linear density (when $L$ and $T$ are constant).

Stewart, Chapter 2.5, #64

The value of $D_3$ for this problem would be 0. Although the problem does attempt to model a real life situation, the student is neither required to create a model or manipulate the model in a manner that cannot be thought of in purely mathematical terms.
Example #3:

A 5-\text{mg} \text{ bolus of dye is injected into a right atrium. The concentration of the dye (in milligrams per liter) is measured in the aorta at one-second intervals as shown in the chart. Estimate the cardiac output.}

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c(t)$</td>
<td>0</td>
<td>0.4</td>
<td>2.8</td>
<td>6.5</td>
<td>9.8</td>
<td>8.9</td>
<td>6.1</td>
<td>4.0</td>
<td>2.3</td>
<td>1.1</td>
<td>0</td>
</tr>
</tbody>
</table>

Stewart, Chapter 5.4, #81

The value of $D_3$ for this problem would be 1. Since the problem requires creating a mathematical model of the heart, a meaningful, real-world model is necessitated. Hence, the problem is given the value of 1.
2.8 $D_4$ - Cognitive Difficulty

This variable measures the cognitive complexity of the problem. The problems are given values according to the following table:

<table>
<thead>
<tr>
<th>Value</th>
<th>Difficulty of the Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Simple recognition and recall or usage of routine procedures-or algorithms in routine ways</td>
</tr>
<tr>
<td>2</td>
<td>Generalizations, explanation, problem solving, proof of a given result</td>
</tr>
<tr>
<td>3</td>
<td>Proofs of a more difficult nature, conjecturing a finding, minor leaps of imagination</td>
</tr>
<tr>
<td>4</td>
<td>Substantial leaps of imagination required</td>
</tr>
</tbody>
</table>

Example #1:

Apply the differentiation rules of this section to find the derivatives of the function $g(t)=1-3t^2-2t^4$.

Edwards and Penney, Chapter 3.2, #2

The value of $D_4$ for this problem would be 1. Because the solution requires the differentiation rules to be applied in a routine manner, it clearly falls under usage of routine procedures in routine ways.

Example #2:

Apply Newton's method to $f(x)=e^{-x}$, starting from $x_0=0$. Explain what happens.

Ostebee Zorn, Chapter 4.4, #16

The value of $D_4$ for this problem would be 2. Because the problem requires an explanation as to why Newton’s Method fails, the problem extends beyond routine procedures and clearly falls under the category having value 2.
Example #3:

Prove that the centroid of any triangle is located at the point of intersection of the medians. C Hints: Place the axes so that the vertices are \((a,0), (0,b), \) and \((c,0)\). Recall that a median is a line segment from a vertex to the midpoint of the opposite side. Recall also that the medians intersect at a point two-thirds of the way from each vertex (along the median) to the opposite side.

Ostebee Zorn, Chapter 4.4, #16

The value of \(D_4\) for this problem would be 3. Although numerous hints are given to the solution of the problem, the question still requires considerable algebraic manipulation to solve. Further, the proof nature of the problem increases its difficulty due to the abstraction required.

Example #4:

Suppose \(f\) is a function that satisfies the equation \(f(x+y)=f(y)+x^2y+xy^2\) for all real numbers \(x\) and \(y\).
Suppose also that \(\lim_{x \to 0} f(x) = 1\).

(a) Find \(f(0)\).
(b) Find \(f'(0)\).
(c) Find \(f'(x)\).

Stewart, Chapter 2.1, #64

The value of \(D_4\) for this problem would be 4 as the solution requires several substantial leaps of imagination. To solve (a), the student must substitute \(x = y = 0\) to find that \(f(0+0)=f(0)+f(0)+0^2?0+0?0^2\) and hence \(f(0)=0\). To solve (b), the student must recognize that \(\lim_{x \to 0} f(x)=1\) is an alternate expression of the definition of the derivative, since at \(x = 0\), \(\lim_{h \to 0} f(x+h)-f(x)=\lim_{h \to 0} f(h)-f(0)=\lim_{h \to 0} f(h)\) which is the given expression with \(x\) substituted for \(h\). Hence, \(f'(0)=1\). To solve (c), the student must recognize that \(f'(x)=\lim_{h \to 0} \frac{f(x+h)-f(x-h)}{2h}\).
\[
\lim_{h \to 0} \frac{f(x)+f(h)+x^2h+xh^2- (f(x)+f(-h)-x^2h-xh^2)}{2h} \\
=\lim_{h \to 0} \frac{2x^2h+f(h)-f(-h)}{2h} = \lim_{h \to 0} \frac{x^2 + f(h)}{h} = x^2 + 1 \text{ since } f(h+(-h)) = f(0) = 0 = f(h) + f(-h).
\]
Hence, the initial substitution and later usage of two different forms of the definition of the derivative place serious demand for creativity on the student, warranting a value of 4.
2.9  $R_1$ - Graphical Representation

This variable identifies the extent to which graphical representations exist in the problem. The problems are given values according to the following table:

<table>
<thead>
<tr>
<th>Value</th>
<th>Graphical Representation of the Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>No interpretation or creation of graphical representations is required</td>
</tr>
<tr>
<td>1</td>
<td>Interpretation but not creation of graphical representations is required</td>
</tr>
<tr>
<td>2</td>
<td>Creation but not interpretation of graphical representations is required</td>
</tr>
<tr>
<td>3</td>
<td>Interpretation and creation of graphical representations is required</td>
</tr>
</tbody>
</table>

Example #1:

Use the limit definition of the derivative to compute $f'(4)$ for the function $f(x) = x^2 - 3x$.

Ostebee Zorn, Chapter 2.6, #61

The value of $R_1$ for this problem would be 0. Because the question nor the solution require the creation or interpretation of a graph, the value 0 is assigned.

Example #2:

Find $f'(x)$ for the function $f(x) = \sin(x) + \ln(x)$. Check that your answer is reasonable by comparing the graphs of $f$ and $f'$.

Stewart, Chapter 3.4, #47

The value of $R_1$ for this problem would be 3. Because the question necessitates both creation and subsequent interpretation of a graph, the value 3 is assigned.
2.10 \( R_2 \) - Numeric or Tabular Representation

This variable identifies the extent to which numeric or tabular representations exist in the problem. The problems are given values according to the following table:

<table>
<thead>
<tr>
<th>Value</th>
<th>Numeric or Tabular Representation of the Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>No interpretation or creation of numeric or tabular representations is required</td>
</tr>
<tr>
<td>1</td>
<td>Interpretation but not creation of numeric or tabular representations is required</td>
</tr>
<tr>
<td>2</td>
<td>Creation but not interpretation of numeric or tabular representations is required</td>
</tr>
<tr>
<td>3</td>
<td>Interpretation and creation of numeric or tabular representations is required</td>
</tr>
</tbody>
</table>

**Example #1:**

The summit of a hill is 100 ft higher than the surrounding level terrain, and each horizontal cross section of the hill is circular. The following table gives the radius \( r \) (in feet) for selected values of height \( h \) (in feet) above the surrounding terrain. Use Simpson's approximation to estimate the volume of the hill.

<table>
<thead>
<tr>
<th>( h )</th>
<th>0</th>
<th>25</th>
<th>50</th>
<th>75</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
<td>60</td>
<td>55</td>
<td>50</td>
<td>35</td>
<td>0</td>
</tr>
</tbody>
</table>

Edwards and Penney, Chapter 6.2, #40

The value of \( R_2 \) for this problem would be 1. Because the question requires the interpretation (but not creation) of tabular data, the value 1 is assigned.

**Example #2:**

Compute \( \Delta y, \ dy, \) and \( \Delta y - dy \) for the given value of \( x \) and for each of the following values of \( dx = \Delta x: 1, 0.5, 0.1, \) and \( 0.01 \) for the function \( y = 2x^3 + 3x - 4 \) and \( x = 3. \)

Stewart, Chapter 2.9, #17
The value of $R_1$ for this problem would be 2. Because the question necessitates creation (but not interpretation) of tabular data, the value 2 is assigned.
2.11 \( R_3 \) - Symbolic Representation

This variable identifies the extent to which symbolic representations exist in the problem. The problems are given values according to the following table:

<table>
<thead>
<tr>
<th>Value</th>
<th>Symbolic Representation of the Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>No interpretation or creation of symbolic representations is required</td>
</tr>
<tr>
<td>1</td>
<td>Interpretation but not creation of symbolic representations is required</td>
</tr>
<tr>
<td>2</td>
<td>Creation but not interpretation of symbolic representations is required</td>
</tr>
<tr>
<td>3</td>
<td>Interpretation and creation of symbolic representations is required</td>
</tr>
</tbody>
</table>

Example #1:

The minute and hour hands on the face of a town clock are 7 feet and 5 feet long, respectively. How fast is the distance between the tips of the hands increasing when the clock reads 9:00?

Ostebee Zorn, Chapter 4.8, #18

The value of \( R_3 \) for this problem would be 2. Because the question requires the creation (but not interpretation) of symbolism, the value 2 is assigned.

Example #2:

Find the limit in the series \( \sum_{k=3}^{\infty} \frac{e^{\sqrt{k}}}{\pi} \)

Ostebee Zorn, Chapter 11.2, #12

The value of \( R_3 \) for this problem would be 3. Because the question necessitates interpretation and creation of symbolism, the value 3 is assigned.
2.12 $R_4$ - Written Representation

This variable identifies the extent to which written representations exist in the problem. The problems are given values according to the following table:

<table>
<thead>
<tr>
<th>Value</th>
<th>Written Representation of the Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>No interpretation or creation of written narratives is required</td>
</tr>
<tr>
<td>1</td>
<td>Interpretation but not creation of written narratives is required</td>
</tr>
<tr>
<td>2</td>
<td>Creation but not interpretation of written narratives is required</td>
</tr>
<tr>
<td>3</td>
<td>Interpretation and creation of written narratives is required</td>
</tr>
</tbody>
</table>

Example #1:

Evaluate the integral $\int 3^x \, dx$.

Edwards and Penney, Chapter 7.4, #25

The value of $R_4$ for this problem would be 0. Because the question nor the solution require the creation or interpretation of a written narrative, the value 0 is assigned.

Example #2:

Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side $L$ if one side of the rectangle lies on the base of the triangle.

Stewart, Chapter 4.7, #19

The value of $R_4$ for this problem would be 1. Because the question necessitates interpretation (but not creation) of a written narrative, the value 1 is assigned.
3. Illustrative Examples and Discussion

Although the examples provided after the definition of each vector component are meant to assist the reader in determining the correct value, they are intentionally limited to the clearer cases. The following examples are meant to provide the reader with a more sophisticated sense of what value should be chosen for more complex examples, as well as the reasoning in choosing the given value.
3.1 C3 - Number of Variables

Verify by direct calculation that if \( k, C, \) and \( d \) are constants, then the function \( P(t) = \frac{C}{1 + d e^{-kCt}} \) is a solution of the logistic DE \( P' = kP(C-P) \).

Ostebee Zorn, Chapter 4.1, #13

The number of variables manipulated, and hence the value of \( C_3 \), for the above problem would be one. Although the variables \( k, C, \) and \( d \) are included in the problem, the problem explicitly states that they should be treated as constants, and hence require no manipulation by the student.

Show that the parametric equations
\[
\begin{align*}
x &= x_1 + (x_2-x_1)t \\
y &= y_1 + (y_2-y_1)t
\end{align*}
\]
where \( 0 \leq t \leq 1 \), describe the line segment that joins the point \( P_1(x_1,y_1) \) and \( P_2(x_2,y_2) \).

Stewart, Chapter 9.1, #27

The number of variables manipulated, and hence the value of \( C_3 \) for the above problem would be two. Although strictly speaking there is only one independent variable, namely \( t \), the student is required to manipulate the relationship between \( x \) and \( y \) as functions of \( t \). Hence, we count the variables \( x \) and \( y \) towards the value of \( C_3 \).

Solve the initial value problem \[
\frac{dy}{dx} = 3x^2y^2 - y^2, \quad y(0) = 1.
\]

Edwards and Penney, Chapter 6.5, #19

The number of variables manipulated, and hence the value of \( C_3 \) for the above problem would be two. Although strictly speaking the variables \( x \) and \( y \) are not dependent, the student is required to manipulate both variables simultaneously to solve the problem.
3.2 $C_4$ - Type of Function

Show that the rate of change of the volume of a sphere with respect to its radius is equal to its surface area.

Stewart, Chapter 2.3, #14

The type of function presented within the question is a cubic polynomial, and hence the value of $C_4$ would be a three. Although no function is given explicitly, the functions $V = \frac{4}{3}\pi r^3$ and $SA = 4\pi r^2$ are given implicitly. Hence, the problem receives the value of three, the maximum value of the two functions mentioned.

Find the dimensions of the rectangle (with sides parallel to the coordinate axes) of maximal area that can be inscribed in the ellipse with equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

Edwards and Penney, Chapter 3.6, #36

Here is an example where the rule that $C_4$ describes the equations given and not those produced is violated. If we were simply categorizing $\frac{x^2}{25} + \frac{y^2}{9} = 1$, the value of $C_4$ would be two. But the implied equation that the student must manipulate is $A=bh$, and hence the value of $C_4$ would be two (independent of $\frac{x^2}{25} + \frac{y^2}{9} = 1$ receiving a two).

Establish the rule $d(x^n) = nx^{n-1} \, dx$.

Edwards and Penney, Chapter 3.6, #36

Although the function $x^n$ could easily be thought of as an abstract function, the value of $C_4$ would not be 10. Rather, since the function $x^n$ is treated as a general polynomial, the value of $C_4$ would be 3.
4. Taxonomy Quick Reference Guide

- **C1** - Text
  - 0 - Other resource
  - 1 - Ostebee Zorn
  - 2 - Stewart
  - 3 - Calc & Mathematica
  - 4 - Sallas and Hille
  - 5 - Edwards and Penney

- **D1** - Context
  - 0 - Out of context
  - 1 - One suggestion
  - 2 - Two suggestions
  - 3 - Three suggestions

- **R1** - Graphical
  - 0 - None
  - 1 - Interpretation
  - 2 - Creation
  - 3 - Both

- **C2** - Subject Matter

- **D2** - Steps
  - 0 - Neither
  - 1 - Either/Both

- **C3** - Number of Vars
  - 0 - Neither
  - 1 - Either/Both

- **C4** - Function Type
  - 0 - Constant
  - 10 - Linear
  - 20 - Quadratic
  - 30 - Polynomial
  - 40 - Rational
  - 50 - Algebraic
  - 60 - Trig/Inverse Trip
  - 70 - Transcendental
  - 80 - Combination
  - 90 - Composition
  - ## - Piecewise
  - ## - Abstract
  - ## - Graphical
  - ## - Tabular
  - ## - Series
  - ## - Miscellaneous
  - ## - Diff Eq
  - ## - Parametrically
  - ## - None

- **D4** - Complexity
  - 1 - Routine
  - 2 - Medium
  - 3 - High
  - 4 - Extremely high
5. Taxonomy Quizzes and Solutions

The following series of “quizzes” and “solutions” are provided not only to allow the reader to gain experience in encoding problems, but also to illustrate additional ideas that we have encountered. Although the “answers” provided are what we consider to be correct, there are alternative answers (some of which are mentioned) for many of the problems.

Try to complete the quizzes with only the Taxonomy Quick Reference Guide found earlier in this document.
5.1 Quiz #1

Create the complete encoding for the following problem:

Find the flaw in the following "proof" that \( I = \frac{1}{2} \int_{-1}^{1} x^2 + 1 \, dx = 0 \):

\[
\begin{align*}
I &= \frac{1}{2} \int_{-1}^{1} x^2 + 1 \, dx \\
&= \frac{1}{2} \int_{1}^{x^2 + 1} \frac{1}{1+x^2} \, du = -I \\
&\text{so } I = 0.
\end{align*}
\]

Ostebee Zorn, Chapter 6.2, #82

<table>
<thead>
<tr>
<th>COMPONENT</th>
<th>VALUE</th>
<th>EXPLANATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>R1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>R2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>R3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>R4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
5.2 Solutions to Quiz#1

The following encoding and explanations attempt to best describe the problem:

Find the flaw in the following "proof" that \( \int_{-1}^{1} \frac{dx}{x^2+1} = 0 \):

\[
\begin{align*}
\int_{-1}^{1} \frac{dx}{x^2+1} &= \int_{-1}^{1} \frac{x^2}{1+x^2} dx = -\int_{1}^{1} \frac{1}{1+u^2} du = -\int_{1}^{1} \frac{du}{1+u^2} \\
\text{so } I &= 0.
\end{align*}
\]

Ostebee Zorn, Chapter 6.2, #82

<table>
<thead>
<tr>
<th>COMPONENT</th>
<th>VALUE</th>
<th>EXPLANATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1 )</td>
<td>1</td>
<td>As defined, Ostebee Zorn problems always receive a value of 1.</td>
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<tr>
<td>( C_2 )</td>
<td></td>
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<tr>
<td>( C_3 )</td>
<td>2</td>
<td>A substitution is made during the &quot;proof&quot;, and hence the student is required to manipulate the relationship between the variables ( x ) and ( u ) when defining the relationship between them, when determining the relationship between ( dx ) and ( du ), and when determining the relationship between the limits of the integrals. Hence, the student manipulates both variables simultaneously, so both should be counted.</td>
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<tr>
<td>( C_4 )</td>
<td>4</td>
<td>Clearly the expression that the student must integrate (as well as all intermediate expressions shown) is a rational function, i.e. a polynomial over a polynomial.</td>
</tr>
<tr>
<td>( D_1 )</td>
<td>1</td>
<td>Since the problem appears in a chapter dealing with ( u )-substitutions, the problem is presented under one notion of context.</td>
</tr>
<tr>
<td>( D_2 )</td>
<td>2</td>
<td>The student must understand the two ideas in the &quot;proof&quot;, namely the initial algebra followed by the change of variables.</td>
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<tr>
<td>$D_3$</td>
<td>0</td>
<td>Clearly, the problem exists strictly within the mathematical realm and includes no real-world modeling or application.</td>
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<tr>
<td>$D_4$</td>
<td>2</td>
<td>The problem extends beyond routine application of the $u$-substitution. At the same time, it does not require any conjecturing or generalization, so it would be given the intermediate value of 2.</td>
</tr>
<tr>
<td>$R_1$</td>
<td>0</td>
<td>Clearly no graphs are created or interpreted by the student.</td>
</tr>
<tr>
<td>$R_2$</td>
<td>0</td>
<td>Clearly no tabular or numerical data is created or interpreted by the student.</td>
</tr>
<tr>
<td>$R_3$</td>
<td>3</td>
<td>The problem necessitates understanding the various steps of the &quot;proof&quot;, which are presented in a symbolic format. Further, the explanation to the flaw of the &quot;proof&quot; will be symbolic in nature. Hence, the student is required to both interpret and create symbolism.</td>
</tr>
<tr>
<td>$R_4$</td>
<td>0</td>
<td>No verbal narrative is created or interpreted by the student. Although the problem necessitates an explanation, it will be symbolic in nature.</td>
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</table>