The Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots$$

where $a \neq 0$

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Circumstances under which the geometric series first appeared

Traditionally, the geometric series is attributed to the Greek Pythagoras; however, it appeared nearly 1500 years prior to in the Rhynd Papyrus. The Rhynd Papyrus was names after a Scottish Egyptologist, A Henry Rhind, who purchased it in 1858 BC in Luxor. The Rhynd Papyrus, measuring 6 metres long and 1/3 metre wide, is one of the oldest mathematical documents in existence, which was written with regards to Egyptian mathematics. Ahmes (1680 BC – 1620 BC), who was the official scribe of the Rhynd Papyrus, copied the document in 1650 BC; however, it was derived from a prototype dating back to 2000 to 1800 BC. The document contains problems and solutions to problems mainly dealing with the areas of fields and volumes of granaries, but other problems do exist. For example the following problem is contained in the Rhynd Papyrus and uses the geometric series:

> There are seven houses; in each house there are 7 cats; each cat kills seven mice; each mouse has eaten 7 grains of barley; each grain would have produced 7 hekat. What is the sum of all the enumerated things?

Even though the Egyptians main usage of the geometric series was to solve problems dealing with the areas of fields and volumes of granaries, the previous problem shows they had other uses for it too!

During sixth century BC Thales and Pythagoras traveled to Babylonia and Egypt to acquire the knowledge of those lands. By obtaining knowledge from the Egyptians Pythagoras led the way for other Greek mathematicians, such as Archimedes, for example, to use the geometric series. The way in which the Greeks used the geometric series was to solve for areas of figures by "exhausting" the area of the given polygon (i.e. method of exhaustion). For example, they would exhaust a circle by cutting it up into squares and triangles. "This is an example of a problem, involving a distinction between approximate and exact computation, that is unlike any considered by the Babylonians and Egyptians" (Edwards 8).

Examples of the uses of the geometric series in calculus and pre-calculus

Hopefully, it is obvious the following examples are just a few of the many that are representative of the geometric series. All of the following examples have been categorized by their respective areas, but are great problems for students in calculus and pre-calculus to attempt and hopefully solve!

1. Physics: The geometric series can be used to determine the radioactive decay for the number of nuclei that decay in a certain period of time.

2. Public Policy: When the government or a private company creates new jobs in an area, these new jobs create additional secondary jobs, which create additional jobs, etc. The relationship between these jobs can be written as a consequence of the geometric series.

2. Consumer application: When someone leases a car the geometric series can be used to show the payment scheduling associated with leasing. One may want to review the outcome of using the geometric series before leasing a car!

4. Lottery: Just in case you hit the lottery big someday, you may want to consider applying the geometric series to find the present value of $1,000,000. You may be surprised.

5. Philosophical mathematics (a discussion of pi): What would pi be like in a different numerical system? Would it ever repeat? Examine the possibility that the number pi were rational, meaning it is capable of being expressed as a fraction of two integers. Then it could be written as a geometric series regardless of its number base. However, Adrien Marie Legendre in 1794 proved that pi is indeed irrational; therefore, it cannot be expressed as a fraction of two integers, hence cannot be expressed in terms of the geometric series, hence cannot have repeating groups in any number system.

6. Medicine: Some drugs are administered to people are eliminated from the body exponentially with the passage of time. The amount of the drug in the body just after the nth injection is given by a geometric series. (Ellis and Gulick 529)

(Note: I found examples 1-5 on the Internet site:
http://www.math.montana.edu/~frankw/ccp/calculus/series/geometric/learn.htm
This site is very resourceful with respect to the geometric series. It is especially useful to use in teaching the geometric series, since you can actually work through the problems and give your answer. Also, the solutions are given at the end of the site)

The geometric series and the development of calculus

When studying the geometric series with respect to the development of calculus, it is important to look at the historical background of the geometric series. As mentioned earlier, the geometric series began with the Egyptians followed by the Greeks; therefore, we will take this as our background and begin our look into the geometric series starting with Archimedes. A problem in ancient mathematics was solving V=1/3Ah. The Greeks knew hoe to sum a finite geometric progression. They used reductio ad absurdum to avoid the formal summation (infinite series). Archimedes solved this equation with the previously mentioned method, rather then taking limits explicitly.

A lot of time passed before any new revelations toward the development of the geometric series developed. During the mid-fourteenth century scholars from Merton College in Oxford attacked the problem of quantifying change. One of the scholars was Richard Swineshead, who was known to the medievals as the Calculator. Swineshead solved a problem that can be generalized as follows: a point moving with constant velocity can be represented by the average velocity during the entire interval is double the initial velocity.

In 1350 Nicole Oresme gave a more general form of the geometric series: $a/k + a/k(1 - 1/k) + \ldots + a/k(1 - 1/k)^n + \ldots = a$, where $k > 0$ and k is included in the positive set of real numbers. The study of the infinite series continued during the 15th and 16th centuries following the ideas of Swineshead and Oresme.

John Wallis (1616-1703) had a great influence on the work that Isaac Newton (1642-1727) later did with the geometric series. One area Wallis dealt with was finding the area under curves. His method of finding area under a hyperbola led Newton to first discover the geometric series. Later Newton used the geometric series to divide polynomials. "Thus was banished forever the 'horror of the infinite' that had impeded the Greeks, and was set loose the torrent of infinite series expansions that were to play a central role in the development and applications of the new calculus" (Edwards 187).

The geometric series and the development of logarithms

Newton discovered the first systematic computations for logarithms in the early seventeenth century. He started in 1667 with the hyperbola $y = 1/\ 1 + x$ as discussed in his Mathematical Papers. Newton divided this function out and then integrated term by term: $A(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$. Hence, $A(1 + x) = \log(1 + x)$, the natural log of $1 + x$. Newton did not specifically refer to $A(1 + x)$ as a logarithm; however, he did recognize its character. From this Newton formed a table of logarithms of integers.

In 1668 Nicolas Mercator (1620-1687) published Logarithmotechnia, which mostly contained a table of common logarithms. The other part of this document is where Mercator found his famous series (previously used by

Newton): $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ for the area under the hyperbola y = 1/ (1 + x). He started solving this by long division of the geometric series. Mercator briefly discussed this in Logarithmotechnia; however, John Wallis (1616-1703) had a more concise version of this principle was in his review of Mercator's work, which was published in Philosophical Transactions in 1668. "In a note by Mercator himself in the Philosophical Transactions of 1668, the logarithms determined by hyperbolic segments are referred to as natural logarithms, and he supplies the factor 0.43429 ($=1/\log_e 10$) for transforming from natural to common logarithms" (Edwards 164).

The geometric series and the ancient method of exhaustion

Archimedes (about 287-212 BC) made the first notable advance in finding the area of regions having curved boundaries. He did this by his ingenious use of the "method of exhaustion", which was based on idea of Eudoxus. With this method, Archimedes found the areas of certain complex regions by inscribing larger and larger polygons of known area in such a region so it would eventually be "exhausted." Even though no formal notation of the "limit" existed, the area of the region that Archimedes determined was the "limit" of the areas of the inscribed figures. "The fact that the modern definition of area stems from Archimedes' method of exhaustion is a tribute to this genius" (Ellis & Gulick 228). The use of the geometric series in the method of exhaustion can be seen since the area of the polygon can be represented in terms of each inscribed figure until the figure is "exhausted."

Works cited

Edwards, C.H. The Historical Development of the Calculus New York: Springer-Verlag, 1979.

Ellis, R. & Gulick, D. HBJ Calculus with Analytic Geometry Orlando: Harcourt Brace Jovanovich, 1989.

Stewart, James. Calculus: Early Transcendentals Pacific Grove, California: Brooks/Cole, 1995.

Note: I also used many various Internet sites for this assignment and bookmarked them on my computer. Therefore, I would be able to recall them, if necessary.