M2T2 EXPLORING QUADRILATERALS

Geometry

State Goal 9: Use geometric methods to analyze, categorize and draw conclusions about points, lines, planes and space.

Statement of Purpose:
The activities in this unit visit quadrilaterals from different perspectives. The first few activities focus on the definitions and properties. Then the activities move to three-dimensional representations using nets, isometric dot paper, and constructions of polyhedrons. These activities emphasize visualization skills, an important goal in late elementary and middle school years. The National Council of Teachers of Mathematics recommends more geometry for students in the middle grades than previously in their new *Principles and Standards for School Mathematics* (p. 211).

The activities here present a problem-solving approach to improving both two-dimensional and three-dimensional visualization skills. Students will use geometric tools to create figures and solve problems. Connections will be made with the real world and the uses of geometry especially quadrilaterals in that world.

Highlights

In this unit, we will begin by asking students to explore properties of quadrilaterals and their diagonals. They will then use these properties to create specific quadrilaterals. Then the participants explore a variety of net diagrams, and extend the two-dimensional designs into the three-dimensional world. Students will analyze figures involving quadrilaterals, draw them, and build polyhedrons. All activities address spatial visualization in a variety of ways.

Connections to the Illinois Learning Standards.

Standard 9.A.—Demonstrate and apply geometric concepts involving points, lines, planes, and space. In this module, participants draw and construct two– and three– dimensional prisms using quadrilaterals..

Standard 9.B.—Identify, describe, classify, and compare relationships using points, lines, planes, and solids. Through exploring the relationships of quadrilaterals and prisms, participants look at relationships both within given quadrilaterals (e.g., diagonals of a rectangle always bisect each other) and among different geometric objects (e.g., showing Euler's Formula with a variety of prisms).

Standard 9.C.—Construct convincing arguments and proofs to solve problems. .The activities in this module give participants the opportunity to make predictions and construct arguments around the properties of quadrilaterals.

Table of Contents

Note: Appendices are printed only on the odd pages. This is done to make photocopying easier. That is, each participant should have a copy of all the odd numbered pages, while the instructors should have a copy of all the pages.

M2T2

Materials

Minimal:

- Paper (grid, construction, blank)
- Pencils (both colored and graphite)
- Straightedge (Ruler)
- Scissors
- Transparent tape
- Compass
- Ruler
- Cereal boxes or other boxes that can be cut by participants, one for each participant. They should be empty.
- Compass (or Mira)
- Protractor or angle ruler

Optimal list includes

- Pattern blocks
- Computer with:
    - Internet connection
    - The Geometer's Sketchpad® or Cabri®

Exploring Quadrilaterals: Sides and Angles

Instructor Page

Tips:
In mathematics, a square is a specialized rectangle because it fulfills all the requirements of a rectangle. However, in everyday life people sometimes need to distinguish between rectangles and squares.

Definitions:

- **Quadrilateral:** A four-sided polygon.
- **Parallelogram:** A quadrilateral whose opposite sides are parallel.
- **Rectangle:** A parallelogram with four right angles.
- **Rhombus:** A parallelogram with four congruent sides.
- **Trapezoid:** A quadrilateral with exactly one pair of parallel sides.
- **Kite:** A quadrilateral with one pair of congruent angles, adjacent sides congruent, and no parallel sides.

The Web site for this activity is
http://www.mste.uiuc.edu/m2t2/geometry/quads.html.
At that address is a display of a "Java Sketchpad" file. Participants will not need the Geometer's Sketchpad program to view this interactive applet. It should be visible in a Java-enabled browser.

However, if Geometer's Sketchpad is available, the original file is available for download at
http://www.mste.uiuc.edu/m2t2/geometry/QUADS1.GSP

If computers are unavailable, use Appendix A.
The completed table is below.

Characteristics of Quadrilaterals

	Relationship of opposite sides	**Relationship of adjacent sides**	**Relationship of angles**
Quadrilateral	Opposite sides vary	Adjacent sides vary	Angle measures vary
Rhombus	Both pairs parallel All sides congruent	Adjacent sides congruent	Opposite angles congruent
Parallelogram	Both pairs parallel Opposite sides congruent	Adjacent sides may or may not be congruent	Opposite angles congruent
Square	Both pairs parallel All sides congruent	Adjacent sides congruent	All angles are right
Rectangle	Both pairs parallel Opposite sides congruent	Adjacent sides may or may not be congruent	All angles are right
Trapezoid	One pair parallel	Adjacent sides vary	Angle measures vary

M2T2

Participant Page

Exploring Quadrilaterals: Sides and Angles

Use the hand out from Appendix A and/or the Web page, www.mste.uiuc.edu/m2t2/geometry/quads.html to explore the properties of these quadrilaterals

- What are some attributes about the sides of each figure?
- How do they relate to one another?
- Are they perpendicular? Parallel? When? Always?
- Are they congruent? Always?
- How do the angles relate to each other?

Place this information down in the table below.

Characteristics of Quadrilaterals			
Name of Polygon	**Relationship of opposite sides**	**Relationship of adjacent sides**	**Relationship of angles**
Quadrilateral			
Rhombus			
Parallelogram			
Square			
Rectangle			
Trapezoid			

Construct (on paper or with Sketchpad) the quadrilaterals that satisfy the list below:

Any Quadrilateral
A Parallelogram
A Rectangle
A Square
A Rhombus
A Trapezoid

M2T2

Instructor Page

Exploring Quadrilaterals: Sides, Angles, and Diagonals

Distribute worksheet on the opposite page, angle rulers or protractors, and rulers. Have the participants complete in groups. Each drawing is the beginning of a quadrilateral. Participants are to finish each shape as labeled. This is a review of common quadrilaterals and properties explored in the previous activity.

Start the diagonal aspect of this activity with the rectangle. Ask participants to draw the two diagonals in the rectangle. What do you notice about the diagonals? Have them measure the diagonals and all related segments. Two important properties that they should eventually see are 1) that the diagonals of the rectangle are equal in length and 2) they bisect each other. Ask participants if this is <u>always</u> the case with rectangles.

Are the diagonals perpendicular (use the protractor to determine this)?

This activity lends itself to the use of computers with geometry software. It is easier to see the persistence of certain relationships among sides, angles, and diagonals as the size and shapes of the quadrilaterals changes. A Java-based view of the diagonals is available at
http://www.mste.uiuc.edu/m2t2/geometry/quadsdiag.html
A Geometer's Sketchpad file on which the Java view is based is available at
http://www.mste.uiuc.edu/m2t2/geometry/quadsdiag.gsp

An additional activity with diagonals is available in Appendix B.

Quadrilaterals and Their Diagonals

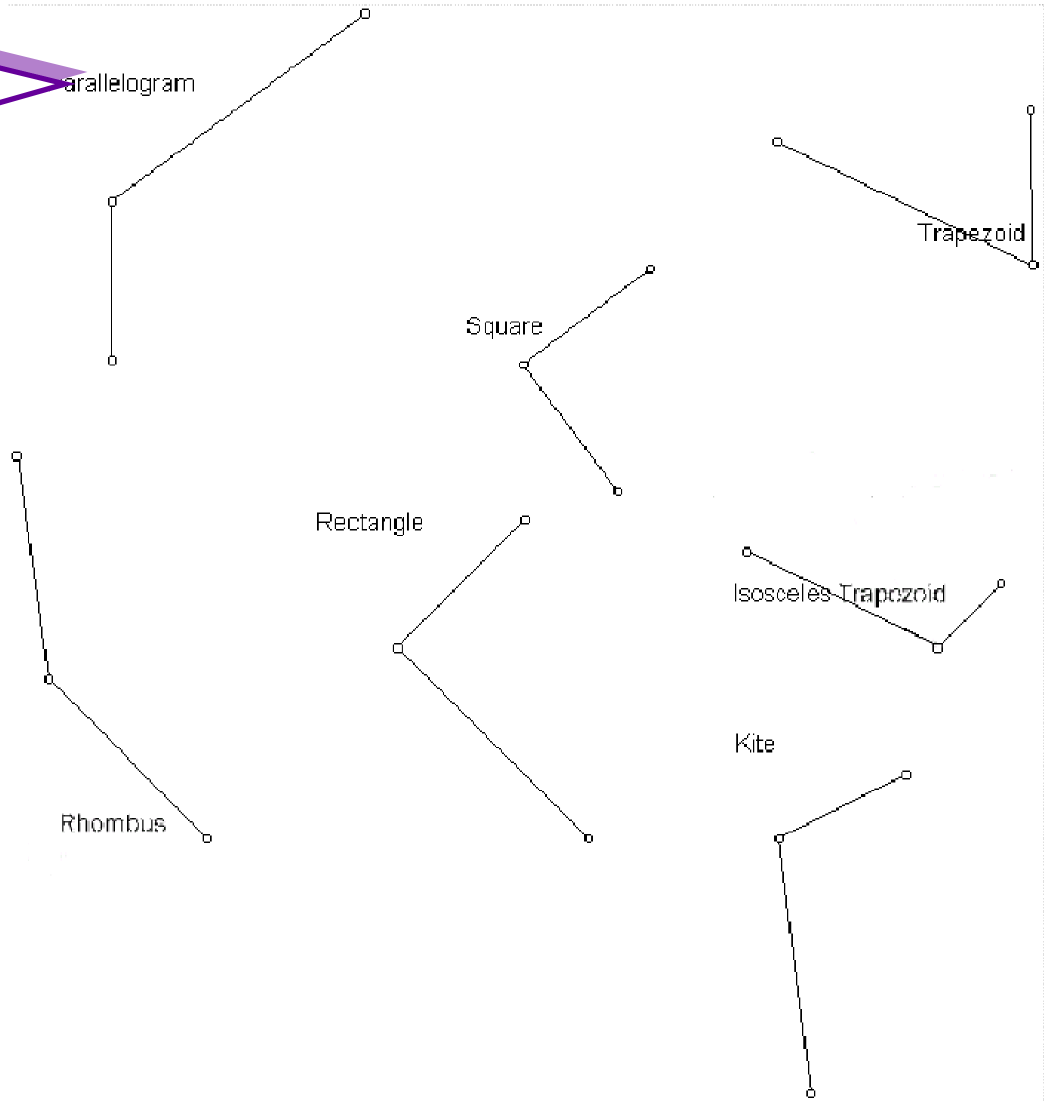
	Diagonals Bisect Each Other?	Diagonals Congruent?	Diagonals Perpendicular?	Diagonals Bisect Opposite Angles?
Quadrilateral				
Rhombus	X		X	X
Parallelogram	X			
Square	X	X	X	X
Rectangle	X	X		
Trapezoid				
Kite			X	X
Isosceles Trapezoid		X		

M2T2

Participant Page

Exploring Quadrilaterals: Sides, Angles, and Diagonals

Each drawing at the right is the beginning of a quadrilateral. Finish each figure. Draw the diagonals, measure them and fill in the table below.



Quadrilaterals and Their Diagonals				
	Diagonals Bisect Each Other?	Diagonals Congruent?	Diagonals Perpendicular?	Diagonals Bisect Opposite Angles?
Quadrilateral				
Rhombus				
Parallelogram				
Square				
Rectangle				
Trapezoid				
Kite				
Isosceles Trapezoid				

Instructor Page

Drawing Quadrilaterals from the Diagonals

Have participants use the page at right to try and draw the quadrilaterals we have been exploring. Each pair of crossed lines are the extension of the diagonals for the quadrilaterals. They are to see if they can place points on those lines in such a way as to make the listed type of quadrilateral.

Participants will find it impossible to draw a few of the quadrilaterals, specifically, the rhombus, the square and the kite. This activity is enhanced by viewing the Java version on the Web at
http://www.mste.uiuc.edu/m2t2/geometry/diags.html
The Sketchpad file for the this is available at
http://www.mste.uiuc.edu/m2t2/geometry/diags.gsp

After completing this worksheet, come up with definitions for all the quadrilaterals discussed so far. A question to consider is whether it is necessary to use the properties of the diagonals in defining the different quadrilaterals.

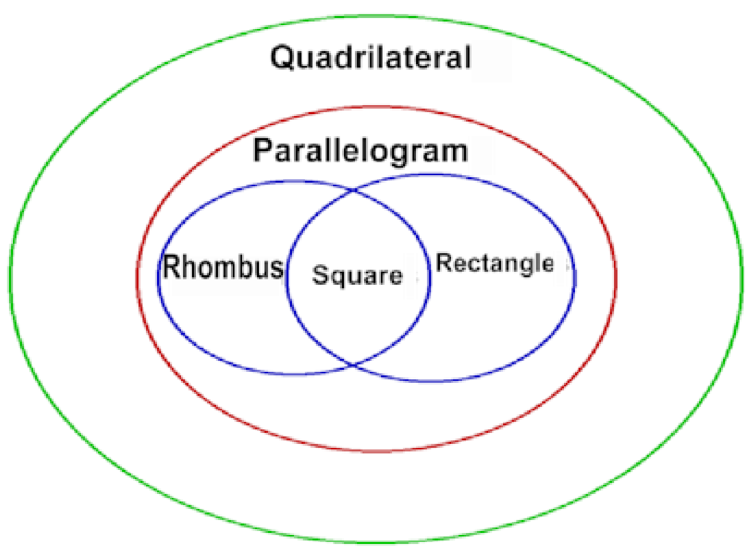
Have participants construct a Venn Diagram like the one below.

This diagrams says, among other things, that

- All the types of polygons we are discussing are quadrilaterals
- All parallelograms are quadrilaterals, but not all quadrilaterals are parallelograms.
- All Rhombi are parallelograms
- Some rhombi are rectangles
- Some rhombi are squares
- Some rectangles are squares
- All rectangles are parallelograms and quadrilaterals
- All squares are rectangles and rhombi and parallelograms and quadrilaterals

Extension:
Have participants write a small quiz based on the properties of quadrilaterals with questions like, "My diagonals are congruent and perpendicular, what kind of quadrilateral could I be?"

Collect the questions into a group quadrilateral quiz.

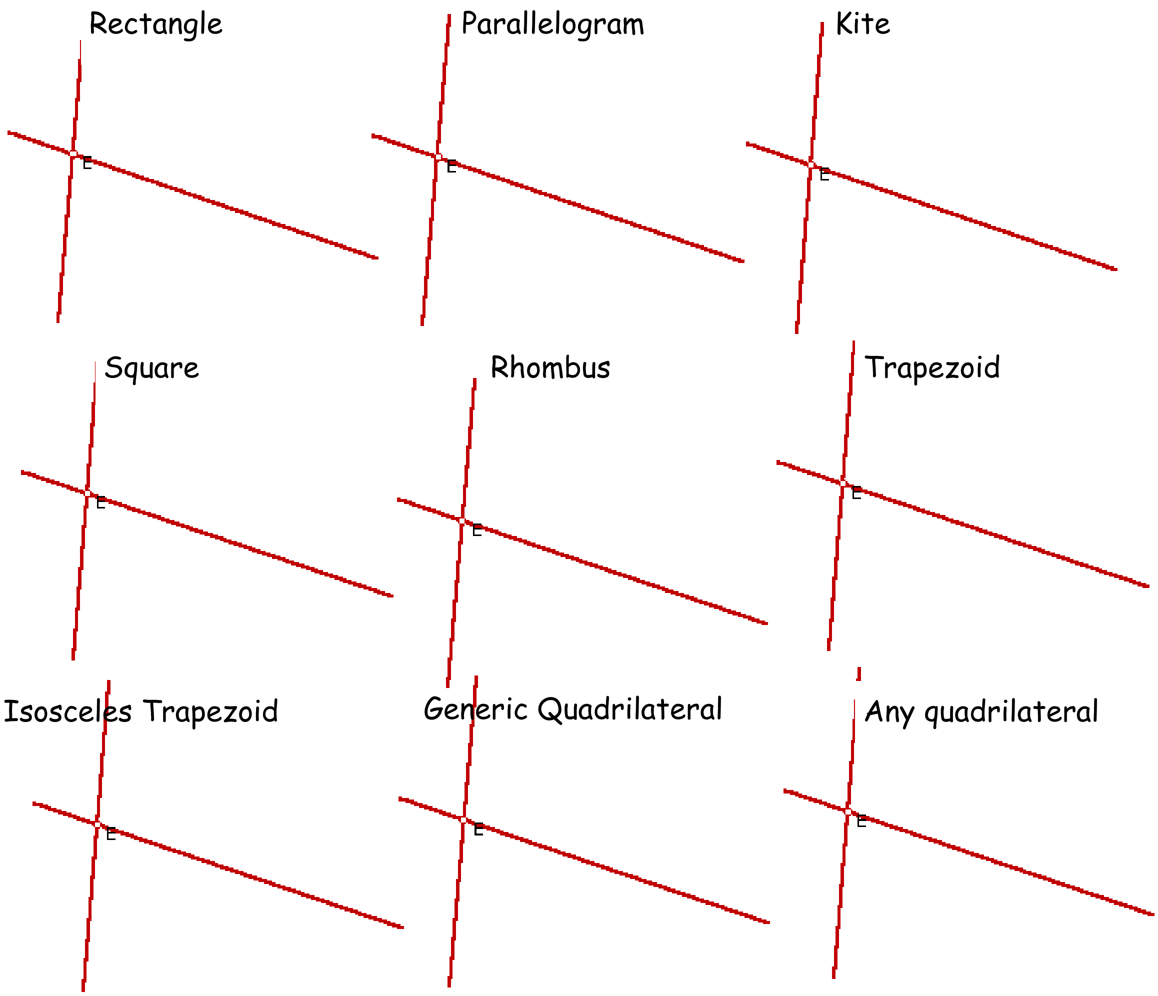


M2T2

Participant Page

Drawing Quadrilaterals from the Diagonals

Place points on the intersecting segments at the left so that they form the vertices of the indicated quadrilateral.



If it is not possible to make the labeled quadrilateral, write "impossible" on the axes.

List the shapes that you could not draw above.	Draw a set of diagonals below that could be used to create these shapes.
________________________________ ________________________________	
Why were you unable to create these shapes?	

M2T2

Instructor Page

Quadrilaterals Within Quadrilaterals

We are going to look at the quadrilaterals formed by connecting the midpoints of quadrilaterals.

The interior quadrilaterals formed by connecting the midpoints may seem obvious for some quadrilaterals (the mid-points of a square form another square), but are sometimes quite surprising. For example, the midpoints of any quadrilateral from a parallelogram! Have participants do the following.

- Draw the midpoint of each of the sides on all the quadrilaterals. (Show them how to do this with a compass, but if they wish to do it by measuring, that may save time.)
- Connect the midpoints of the adjacent sides in each quadrilateral to form another quadrilateral.
- Then have them draw at least two versions of the various quadrilaterals on a separate sheet of paper and find the midpoints and connect them. You may show them the compass and straight-edge constructions if you know them. Or you may have them approximate the drawings as best they can. If you have access to Geometer's Sketchpad or other geometry software, they can use the constructions in the program to make the quadrilaterals.

Participants may find it useful to explore the midpoints with the Java activity at
http://www.mste.uiuc.edu/m2t2/geometry/quadsInQuads.html
The Sketchpad file for this is available at
http://www.mste.uiuc.edu/m2t2/geometry/QUADS2.gsp

Point out to participants that a more in-depth discussion of this topic would lead quickly to the notion of proof. It is not obvious that the midpoints of a rhombus will give you a rectangle, so how would you "prove it"?

If participants are teachers who have had geometry before, you might ask them to try and prove that the shape inside the rhombus must be a rectangle.

Quadrilaterals Within Quadrilaterals

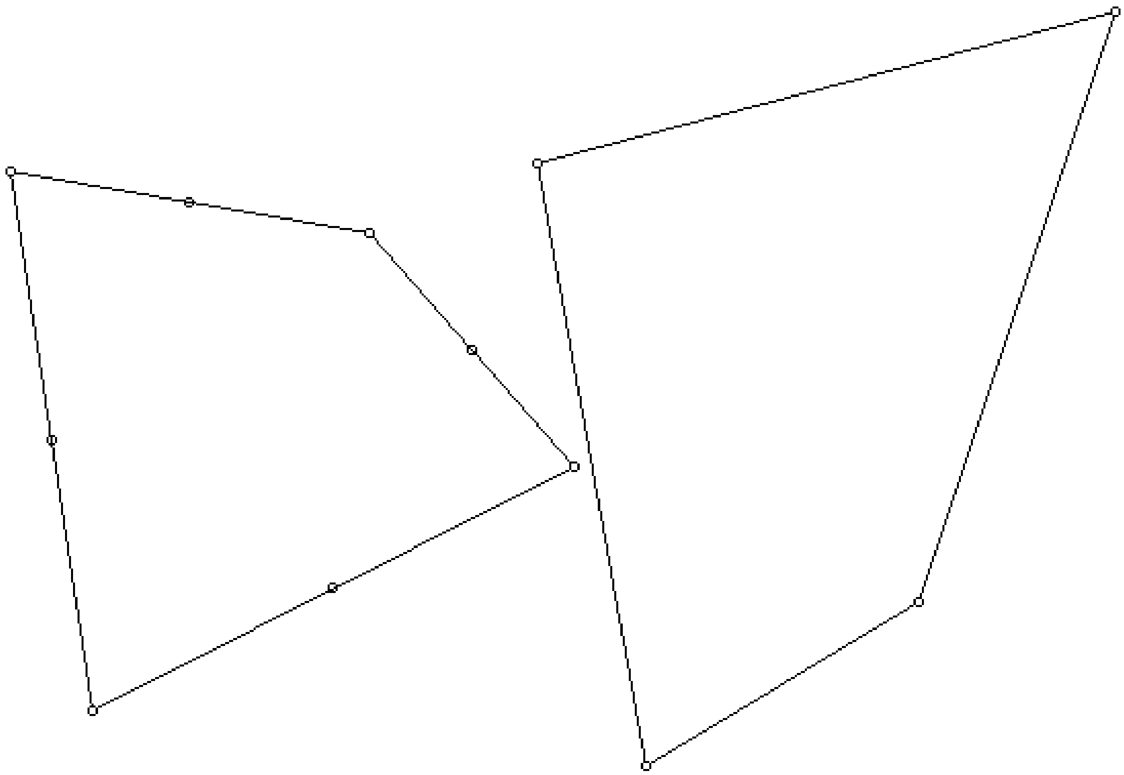
Original Polygon	**Most specific name for the quadrilateral formed by connecting the midpoints**
Quadrilateral	Parallelogram
Parallelogram	Parallelogram
Rectangle	Rhombus
Square	Square
Rhombus	Rectangle
Trapezoid	Parallelogram
Isosceles Trapezoid	Rhombus

A quick proof of the fact that the quadrilateral formed by joining the midpoints of a rhombus is a rectangle is available at http://www.mste.uiuc.edu/m2t2/geometry/Rhmrect.htm.

Participant Page

Quadrilaterals Within Quadrilaterals

Below are two quadrilaterals. One has the midpoints placed on the sides. If you connect those midpoints, you will have drawn another quadrilateral. Make a prediction as to what kind of quadrilateral it will be. Connect the midpoints and measure the sides and angles of quadrilateral formed to see if you were right. Find the midpoints on the other quadrilateral and connect them.



Quadrilaterals Within Quadrilaterals	
Original Polygon	**Most specific name for the quadrilateral formed by connecting the midpoints**
Quadrilateral	
Parallelogram	
Rectangle	
Square	
Rhombus	
Trapezoid	
Isosceles Trapezoid	

On a separate sheet of paper, draw at least two of each of the polygons listed in the table at left. Then find the midpoints of the sides and connect them to find the quadrilateral formed.

Instructor Page

Exploring Quadrilaterals in Three Dimensions

Cutting up cereal boxes
Pass out empty boxes of cereal. Have the participants cut their boxes according to the instructions on the participant sheet.

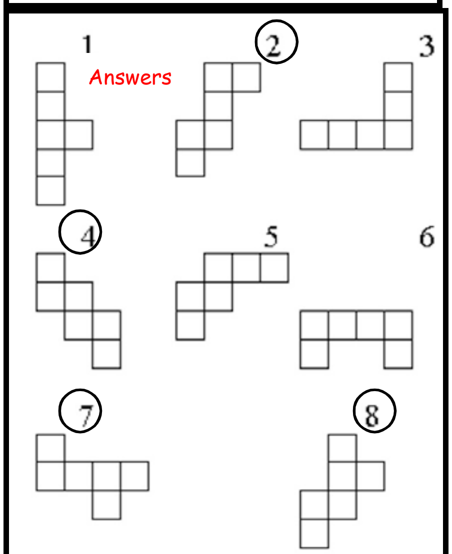
They should sketch their nets and share the nets and boxes with another group to confirm their work.
Can the same box have more than one net? ___YES__
In a cereal box, which sides must be the same sized rectangles? Opposite sides, top and bottom and side panels.

Pass out the net for the cube in Appendix C.
Have participants cut out the net and fold it to form the cube.

Appendix D has graph paper that may help with sketching nets.

Pass out the set of nets in Appendix E, and have participants pick which nets they think will fold to form a cube. It is fine for them to cut the nets out and fold them to check their predictions. The answers are below.
A cube has 8 vertices, 12 edges, and 6 faces.

Circled numbers go with those nets that will form cubes

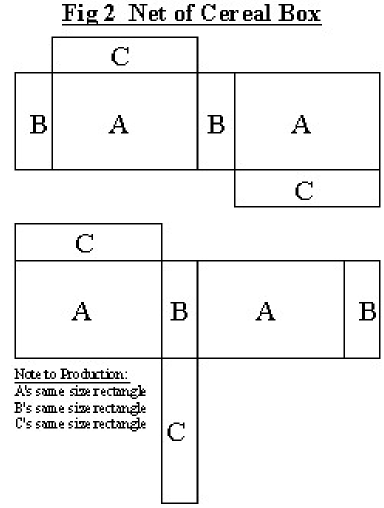


Terms to know:
Face: *Sides of a polyhedron*
Edge: *Segment where two faces meet*
Vertices: *The points where three or more edges meet*

Extension 1

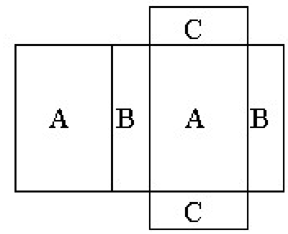
- Have participants cut out the six pieces of their cereal box net.
- Have them find as many arrangements of the six pieces as they can that will fold back into the original box.
- When they find an arrangement, they should sketch the net of this arrangement.
- Have students imagine that these nets were to be placed on a sheet of cardboard to be cut and folded as cereal boxes. Determine which of their nets would require the least amount of cardboard waste when cut out.

Fig 2 Net of Cereal Box



Exploring Quadrilaterals in Three Dimensions

Fig 1 Net of Cereal Box



Cutting up cereal boxes

Your instructor will give you a cereal box. Think of how to cut it so that it will lay flat in one piece on a table top. Then cut the box along the edges and lay it out flat on the table top. If you make mistakes, you can correct them with tape.

In the space below, make a sketch of the flattened box, including the folds. It will look something like the figure above. This sketch is called a **net**. A net is a pattern for a three-dimensional figure. If you cut out the net and fold on the edges, it will form a solid.

Label your net and your box and give them to a different group to confirm that the unfolded box does match the net that was drawn. If it does not, correct the net.

Can the same box have more than one net? _______________

In a cereal box, which sides must be the same sized rectangles?___________________________

You instructor will pass out the net for a cube. Cut it out and fold it together to make the cube. Count the vertices, edges, and faces and put the values in the lines below.

vertices on a cube_______
of edges on a cube_______
# of faces on a cube________

M2T2 Instructor Page

Visualizing Cubes and Stacks of Cubes

Discussion: Each section of the Sears Tower is a tower itself. We could build a model of the Sears Tower using blocks. Ask participants to think about how they could draw a picture of a building and have it appear three-dimensional. It is often helpful to use a special paper to draw such sketches. It is called isometric dot paper.

This activity should take between 10 minutes and half an hour.

Blank sheets of isometric dot paper are available in Appendix F.

⇒ Use overhead copy of the isometric dot paper to sketch one cube and have participants try it themselves. Note: The cube needs to be oriented with one vertical edge centered and the sketch should show the horizontal edges as segments on the diagonal, all parallel..

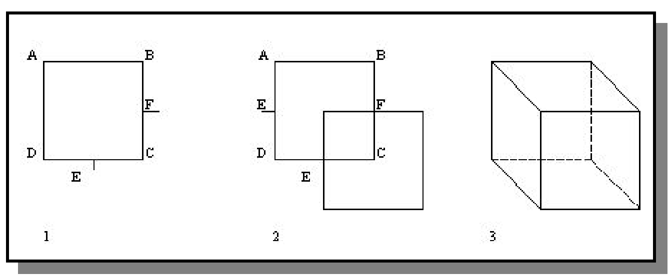
⇒ Give participants about 10 minutes to attempt to replicate the drawings on the isometric dot paper. They may build the figure with blocks to assist in the drawing..

Challenge (Find the pattern):

⇒ Give participants an extra sheet of isometric dot paper and have them draw one cube, then a cube made up of two smaller cubes on each edge. How many cubes are hidden from view? Make a 3 x 3 x 3 cube. How many are hidden from view? In a 4 x 4 x 4? A 5 x 5 x 5? Keep track of the numbers that are hidden in a table like the one below. The answers are in red. Predict how many will be hidden in the drawing of a 50 x 50 x 50 cube.

Be careful not to count the same cube twice!

Number of cubes on each side	1	2	3	4	5	50
Total number of cubes	1	8	27	64	125	50^3
Number of cubes hidden from view	0	1	8	27	64	49^3



To Draw a Cube

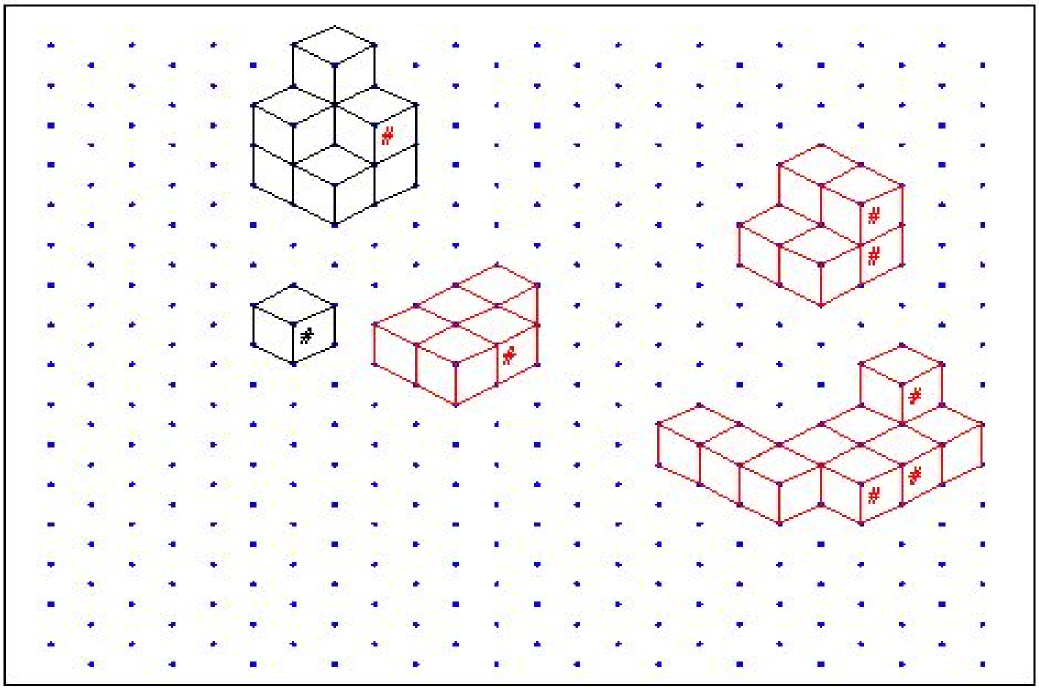
Drawing a cube on regular paper is easy.

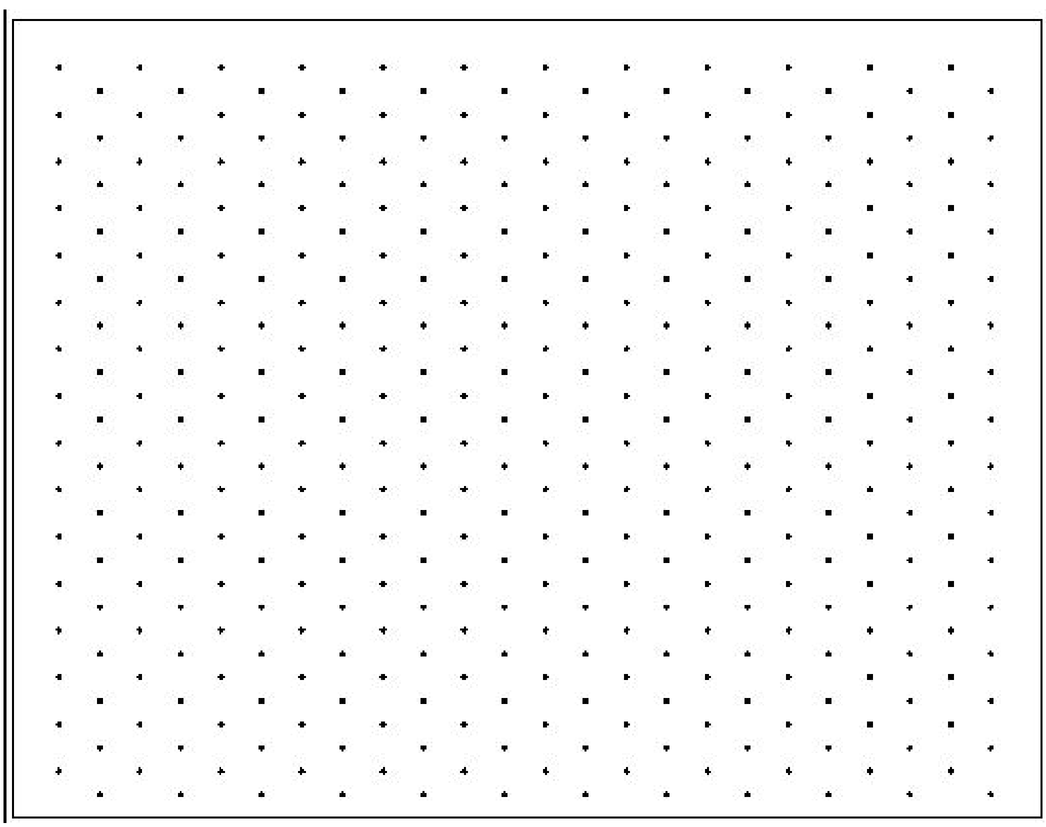
1. Draw a square
2. Draw a second square the same size as the first which overlaps the first at the lower left. The midpoints of the sides of the second square match the midpoints of the sides of the first square (points E and F).
3. Connect the vertices. Hidden lines should be dashed.

Participant Page

Visualizing Cubes and Stacks of Cubes

Below are drawings of several stacks of cubes. Use a **pencil** and the isometric dot paper below to draw the stacks without the cubes that have stars.





Instructor Page

Euler's formula

Larger versions of the nets on the participant page are in Appendix G. Have participants cut out the nets and fold them into polyhedrons.

Have participants create a table like the one below and fill in the values by counting the faces, vertices, and edges for each polyhedron.

Euler's Formula tells us that, for any polyhedron, there is a relationship between the number of faces, edges, and vertices. We can see this by looking at the table below.

One way to write the formula is $V + F - E = 2$. In other words, the number of vertices plus the number of faces minus the edges is always 2.

Notice in the table that the numbers for both triangular prisms are the same. It does not matter that they look somewhat different. Also, the cube is merely a special case of the rectangular prism. So the number of faces, edges and vertices are the same for both.

Illustration of Euler's Formula			
Name of Polyhedron	**# of Faces**	**# of Vertices**	**# of Edges**
A. Tetrahedron	4	4	6
B. Hexagonal Prism	8	12	18
C. Triangular Prism	5	6	9
D. Rectangular Prism (Box)	6	8	12
E. Triangular Prism	5	6	9
Cube	6	8	12

Internet Resource:
A more general view of the "Euler Relationship" can be found on the NCTM Student Math Notes site at http://www.nctm.org/publications/smn/
There is a .pdf file called "Let's Be Discrete" at
http://www.nctm.org/publications/smn/2000-11smn.pdf (requires Adobe Acrobat Reader)

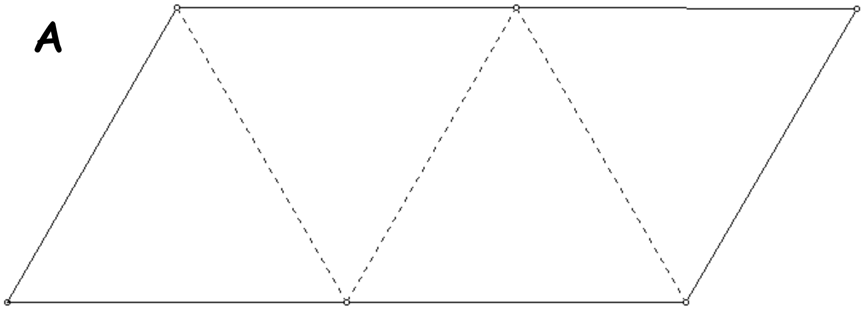
Participant Page

Euler's Formula

Cut out the nets on this page or the larger ones that your instructor gives you. All but one have quadrilaterals for faces. We include a tetrahedron as an illustration of the fact that Euler's Formula holds for all polyhedrons.

Count the Vertices, Faces and Edges of each polyhedron. Put the values in a table and look for a pattern.

A



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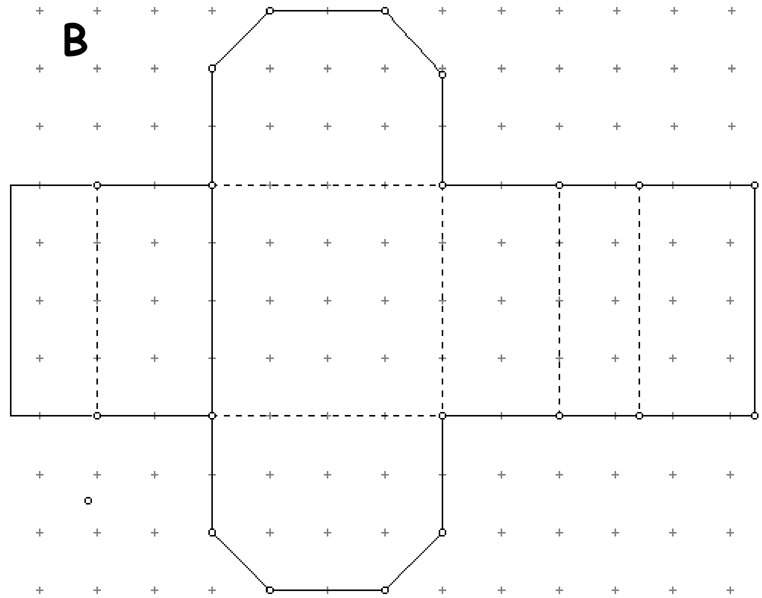

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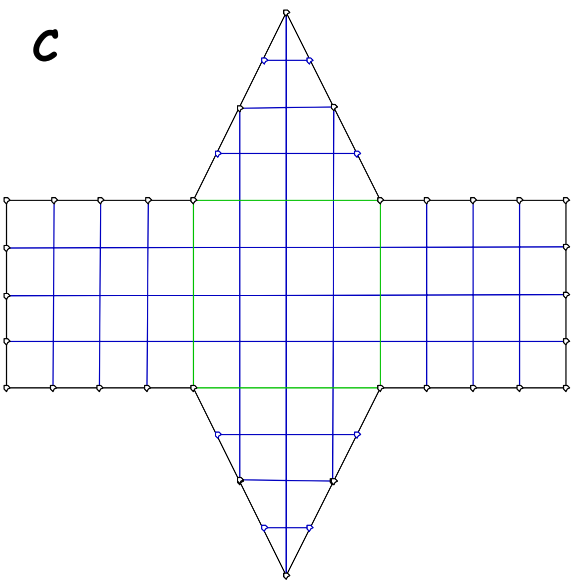

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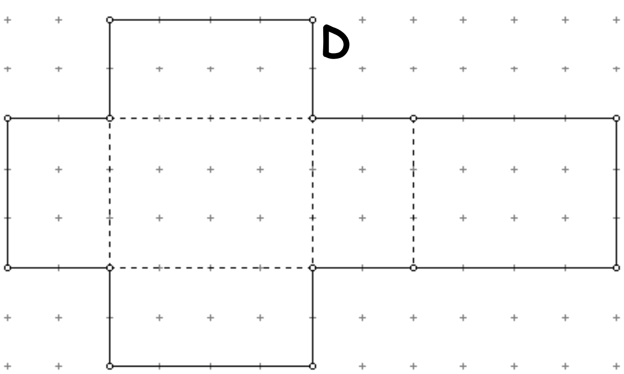

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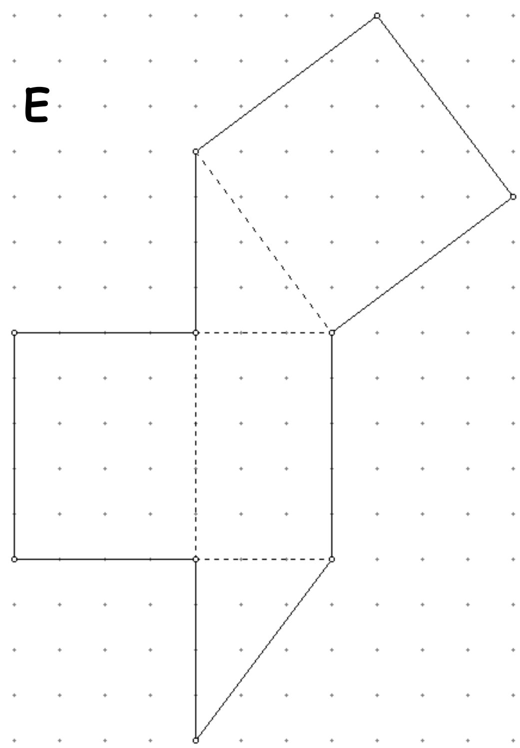

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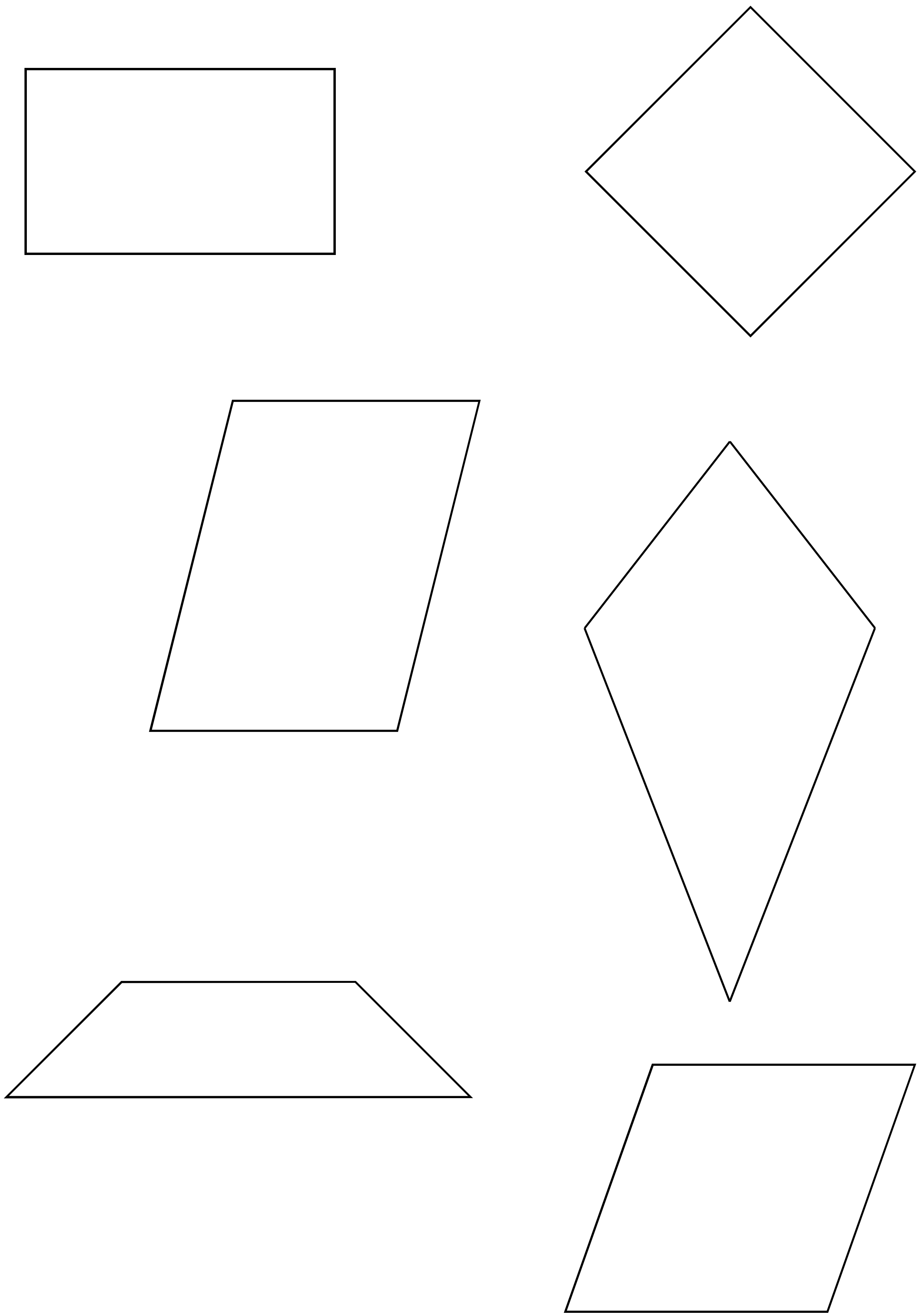
Appendix A

Various Quadrilaterals



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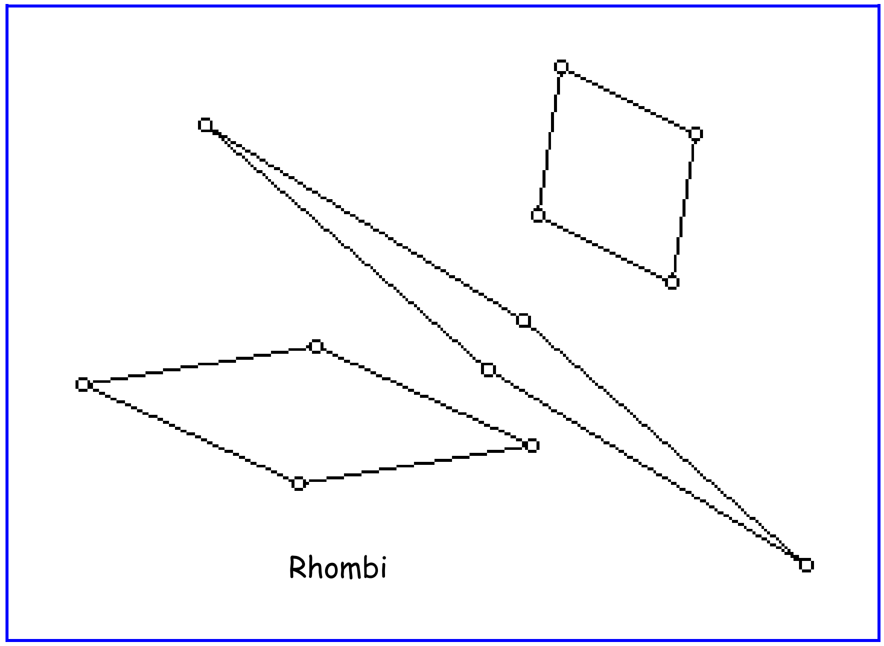
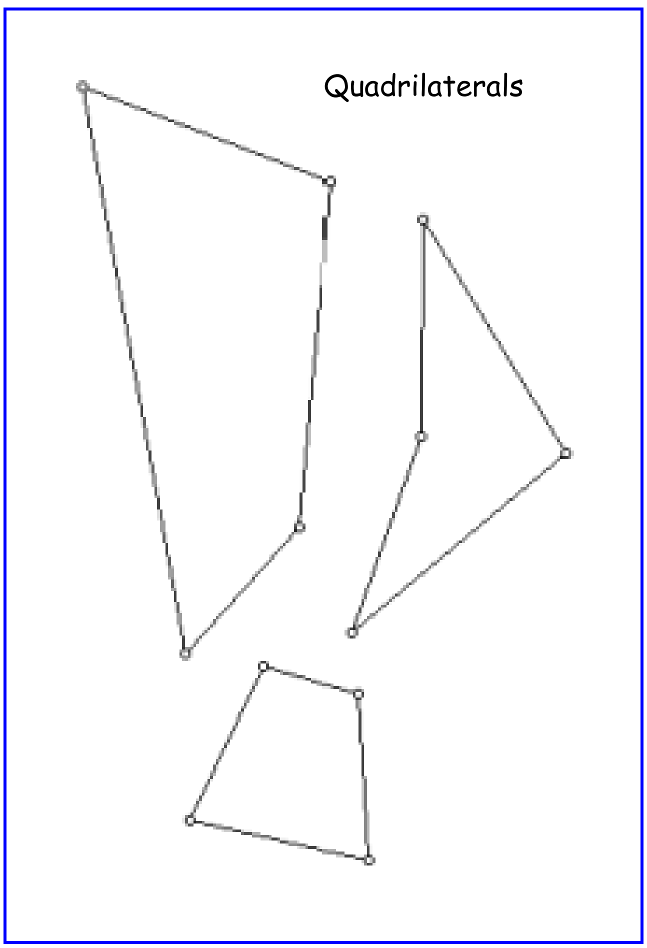
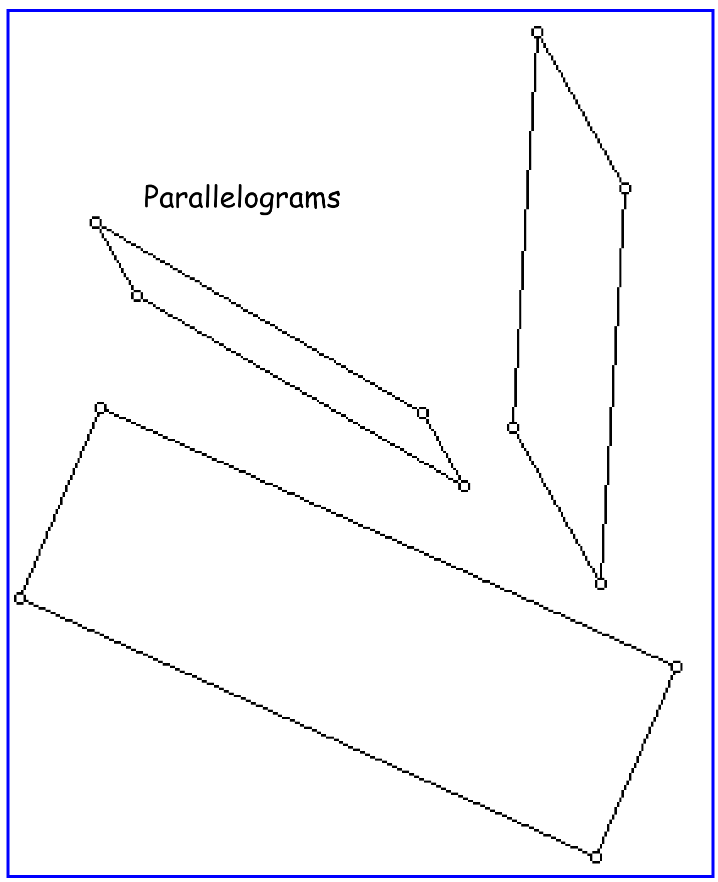
Appendix A (continued)







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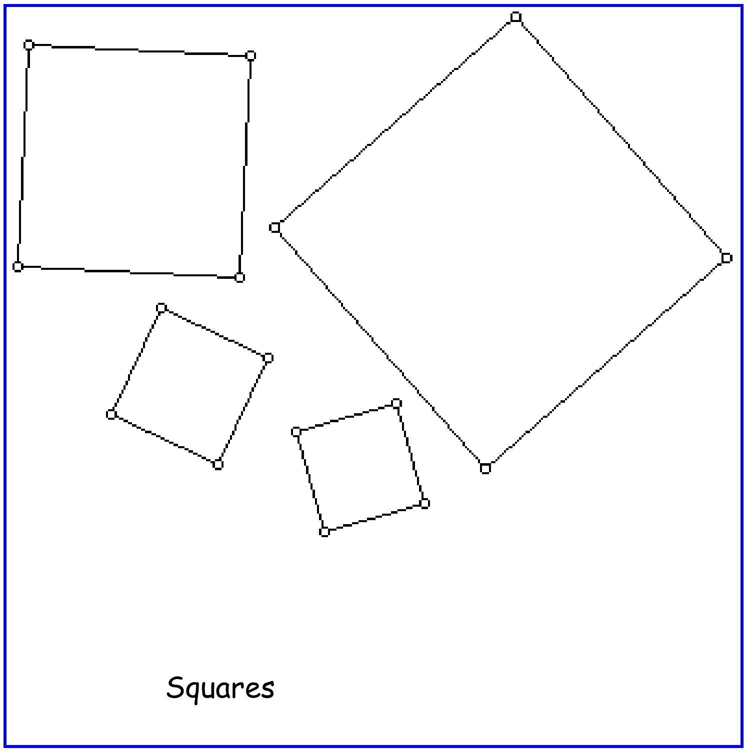
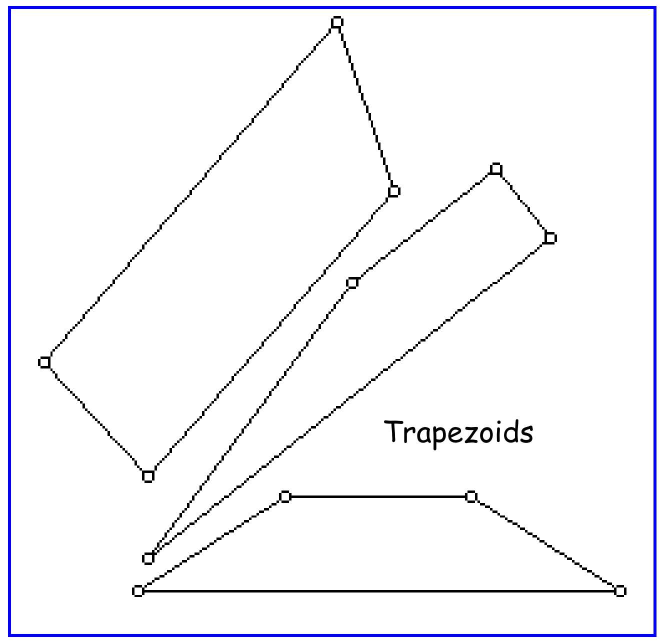
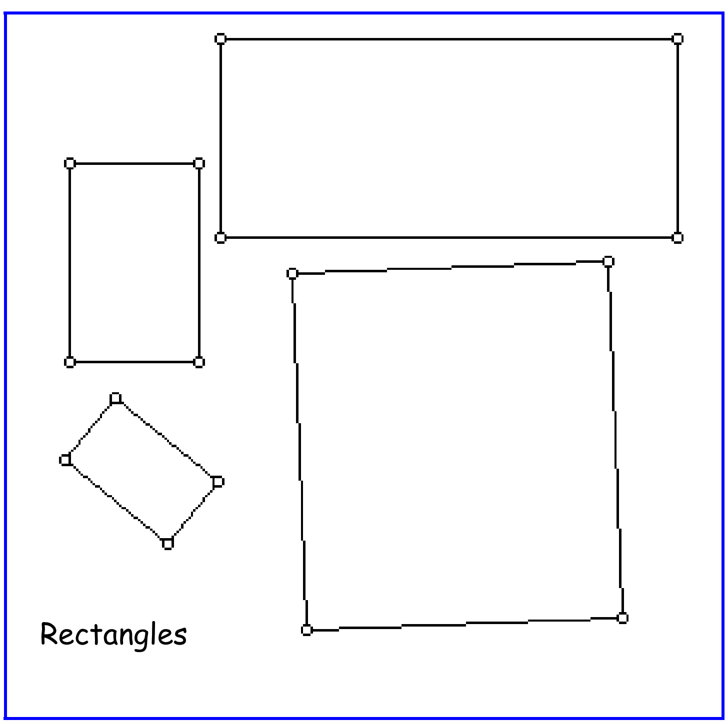
Appendix A (continued)







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Appendix B

Quadrilateral Diagonals

Below are a set of diagonals for quadrilaterals Draw the sides of the quadrilaterals and fill in the table below with the most specific name for the quadrilateral.

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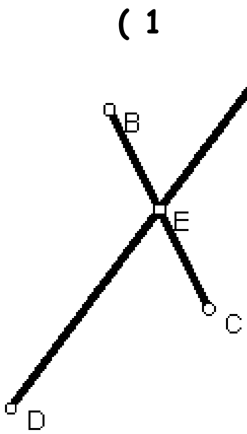

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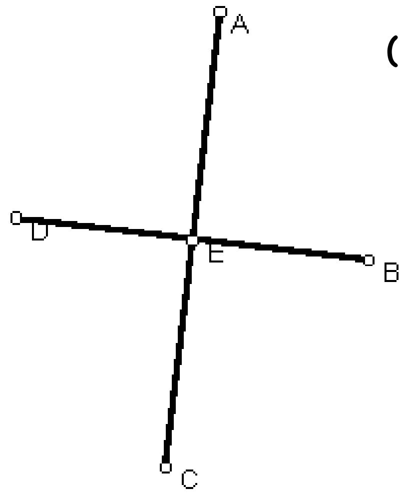

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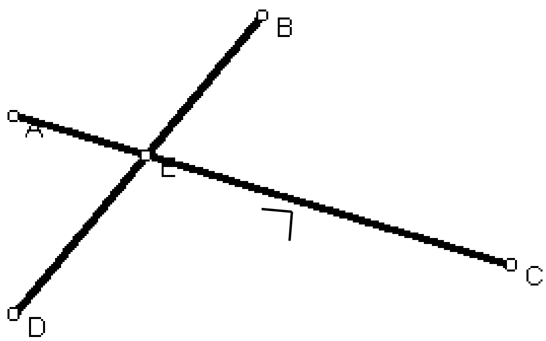

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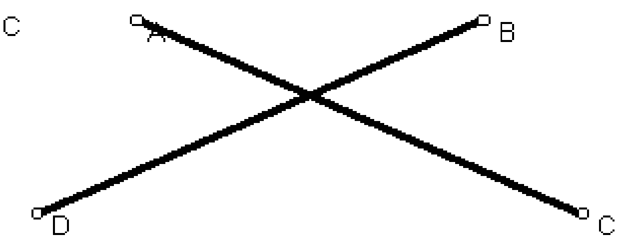

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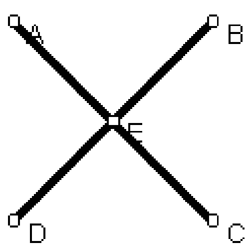

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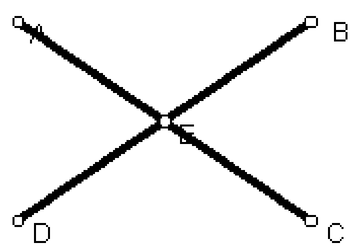

1	**2**
3	**4**
5	**6**

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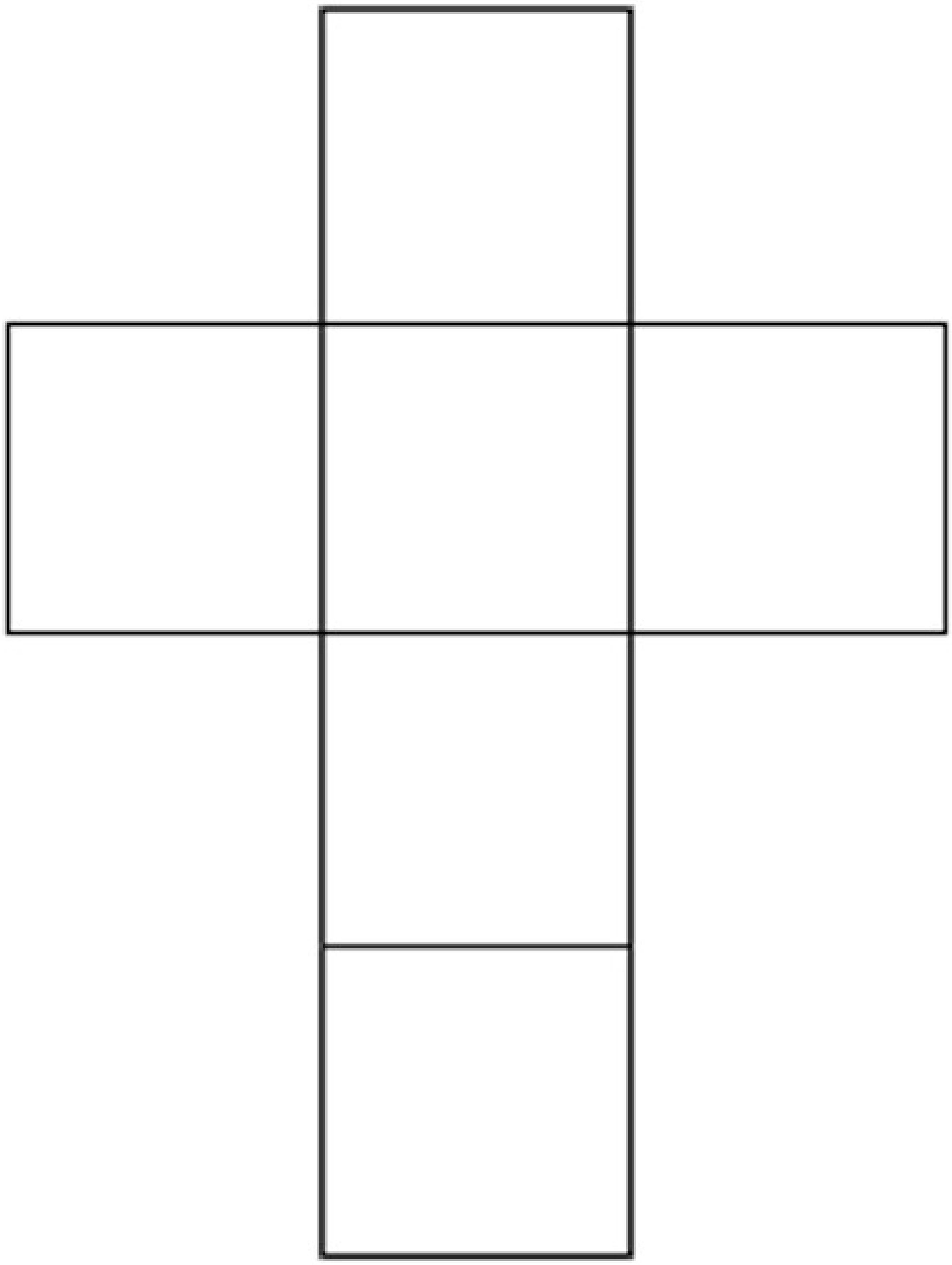
Appendix C

Net for a Cube



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Appendix D

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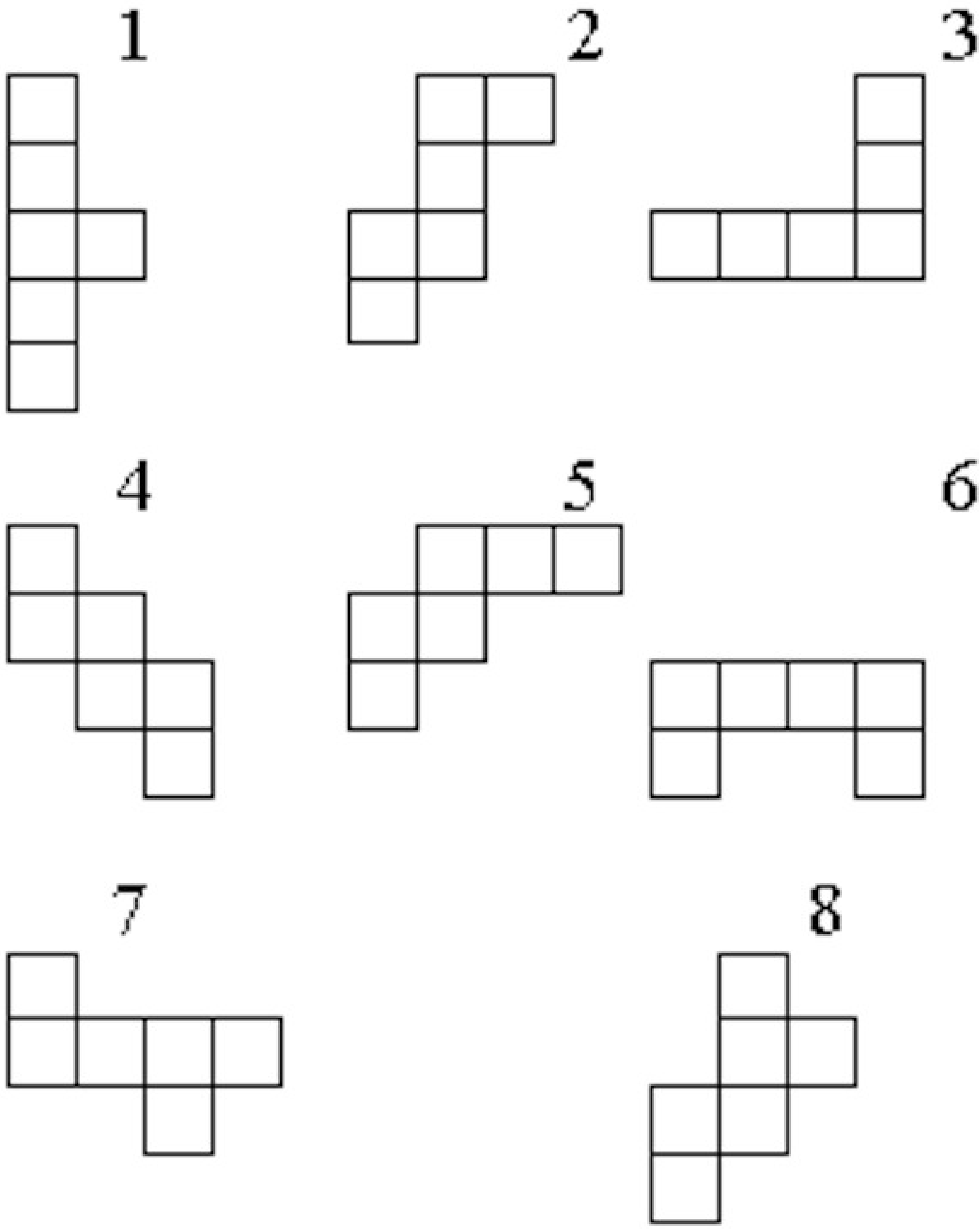
Appendix E

Which Nets Fold to a Cube?

Circle the number of each net that can be folded to form a cube. Check your predictions by cutting out the larger versions of the nets and folding them to see if they form cubes.



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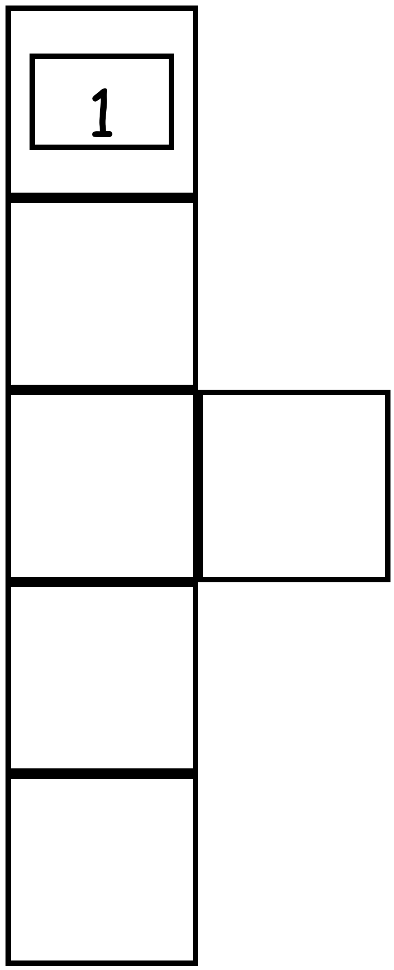
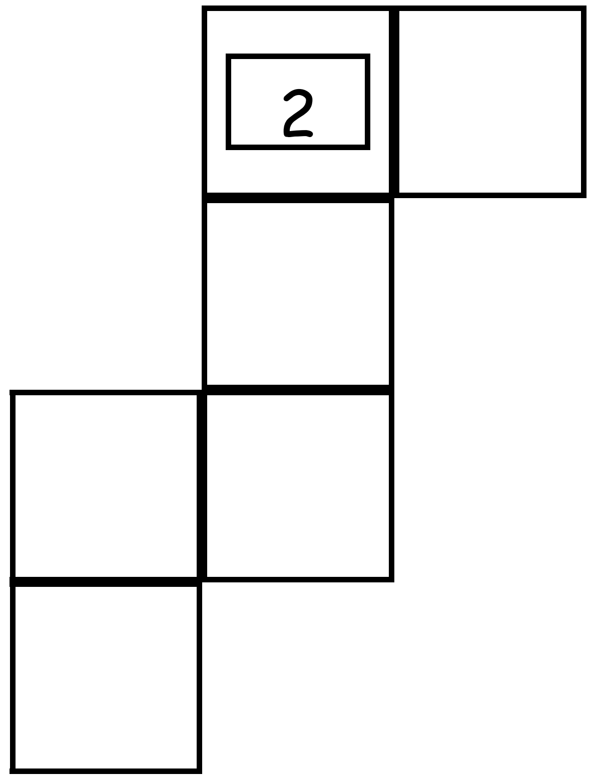
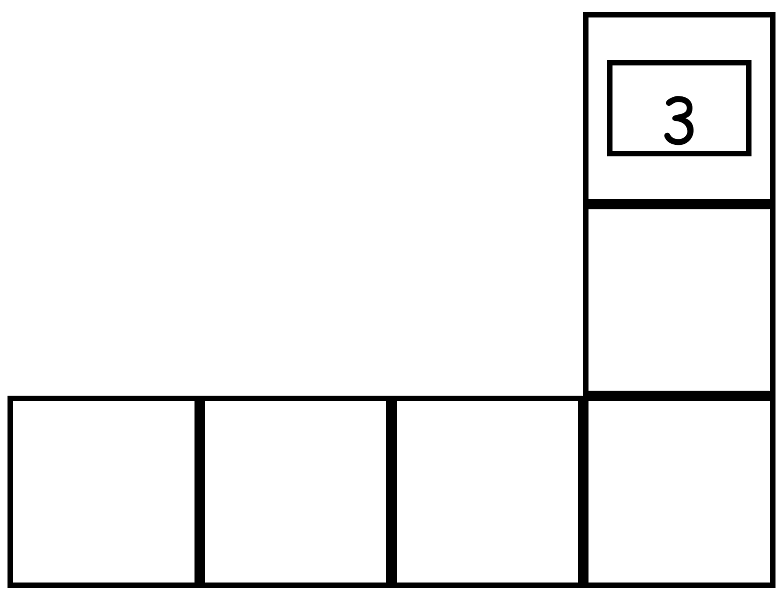
Appendix E (continued)
Larger versions of the Nets







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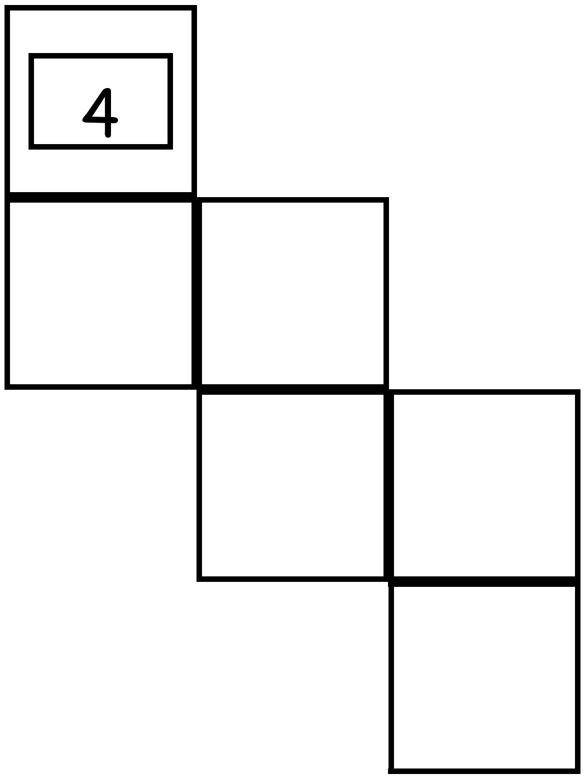
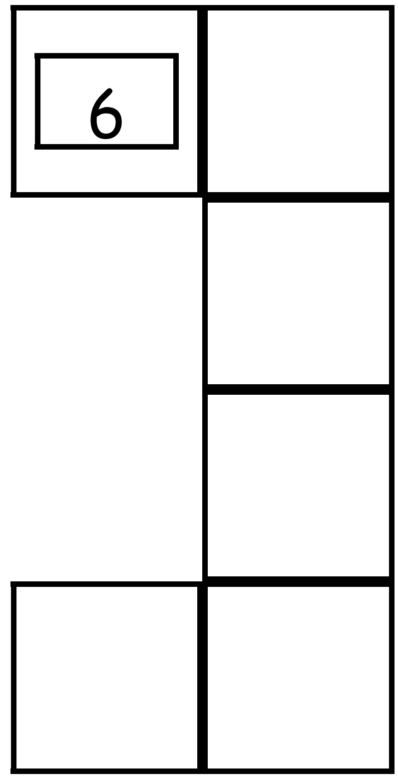
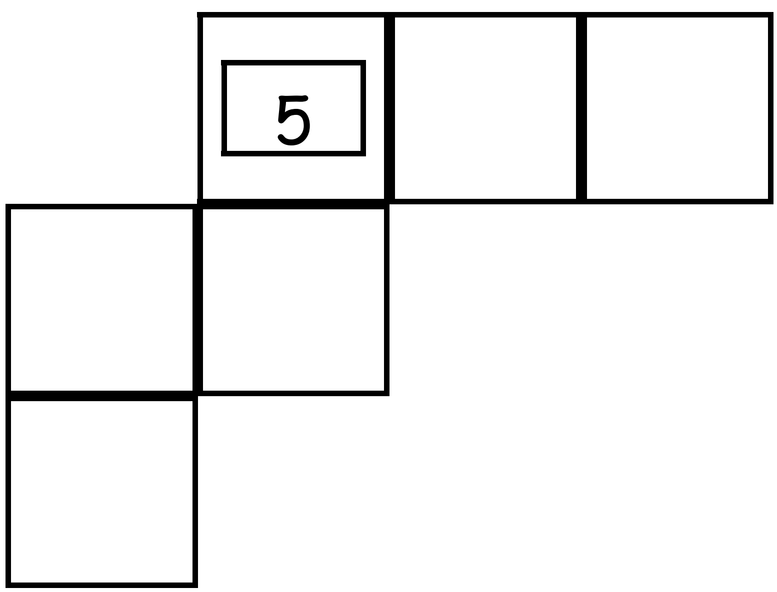
Appendix E
(continued)







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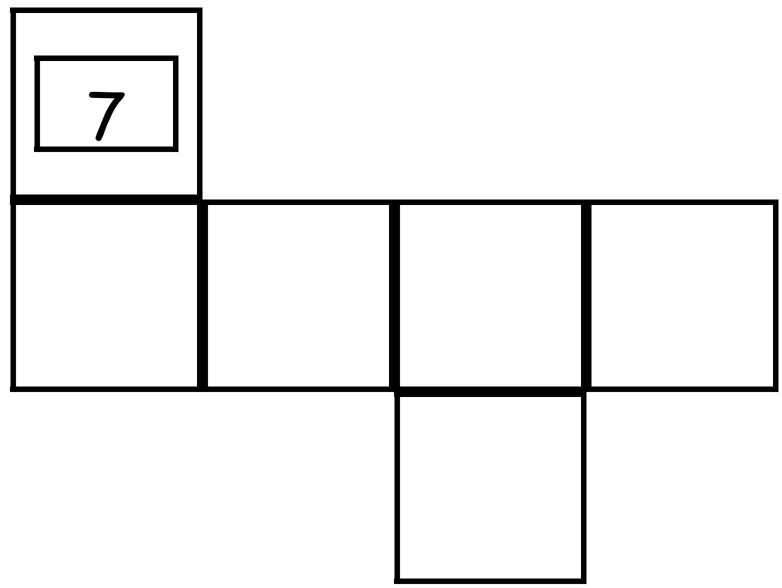
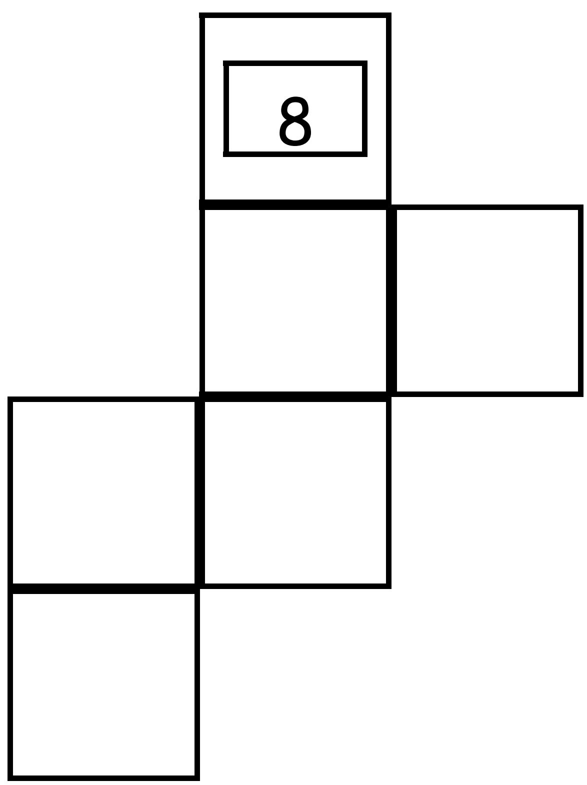
Appendix E
(continued)





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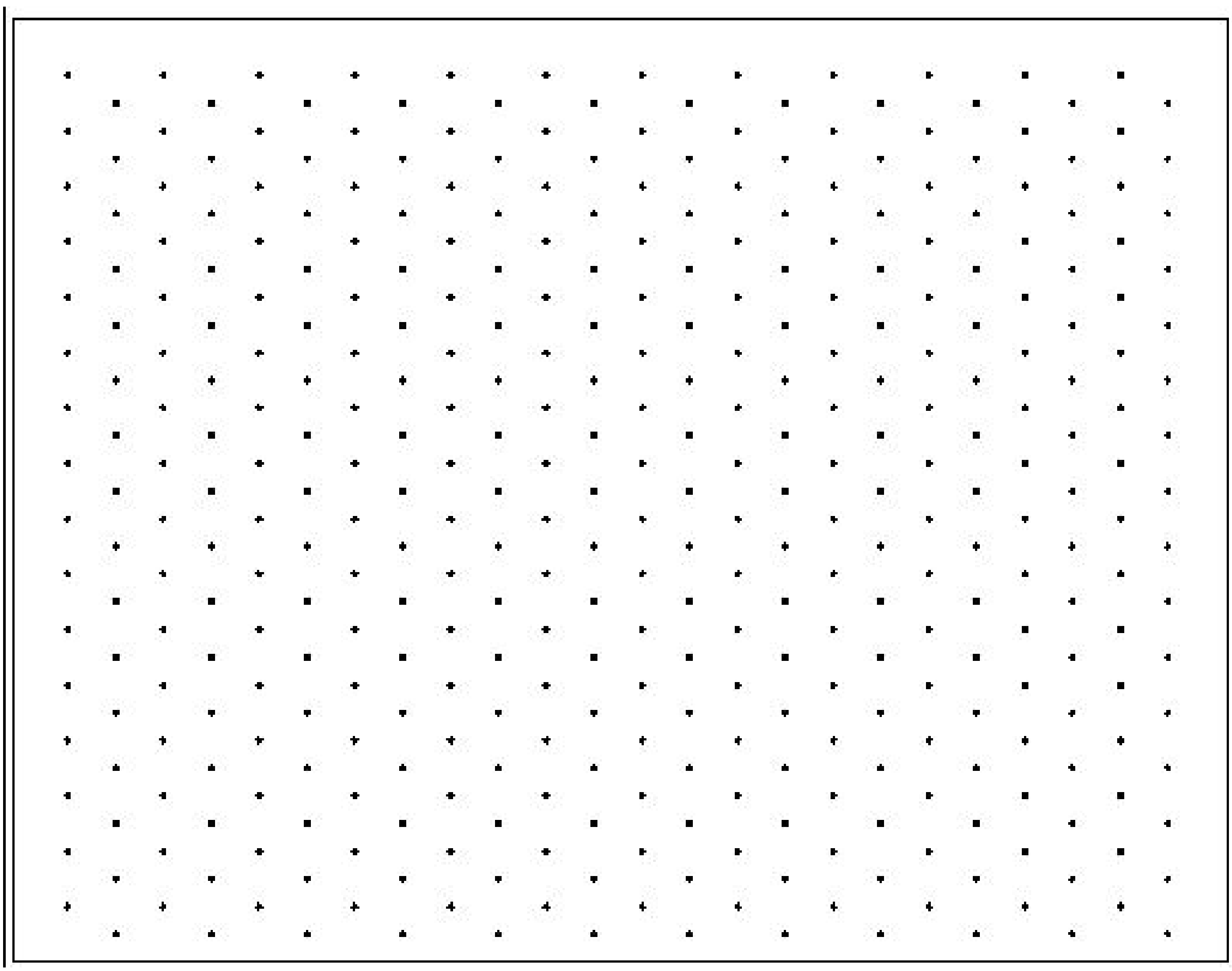
Appendix F

Isometric Dot Paper



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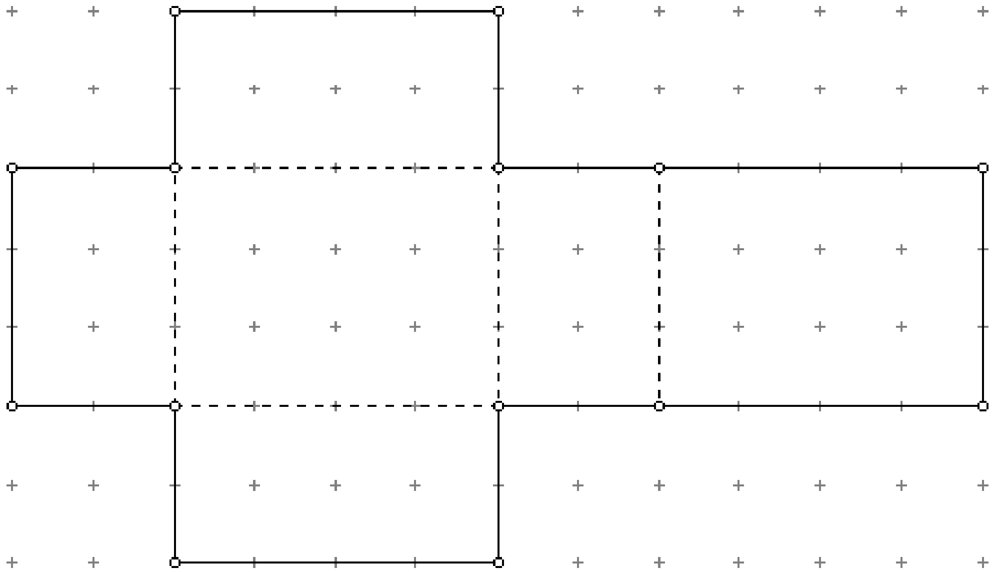
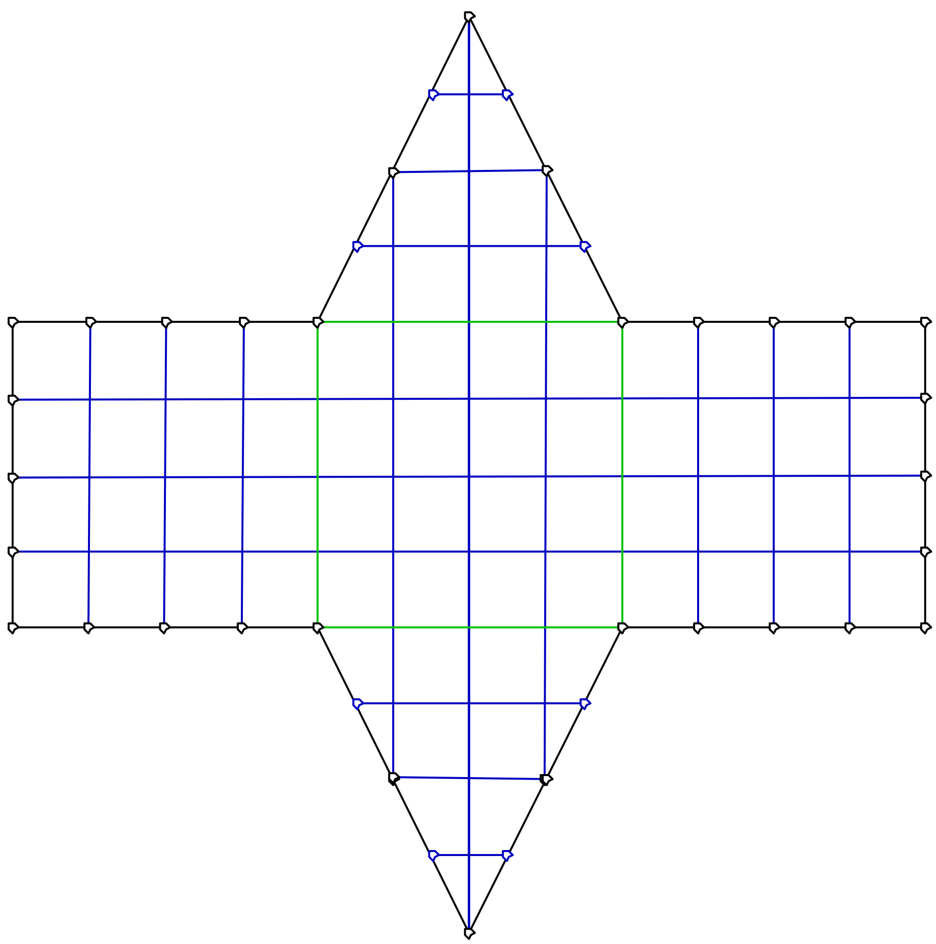
Appendix G





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Appendix G
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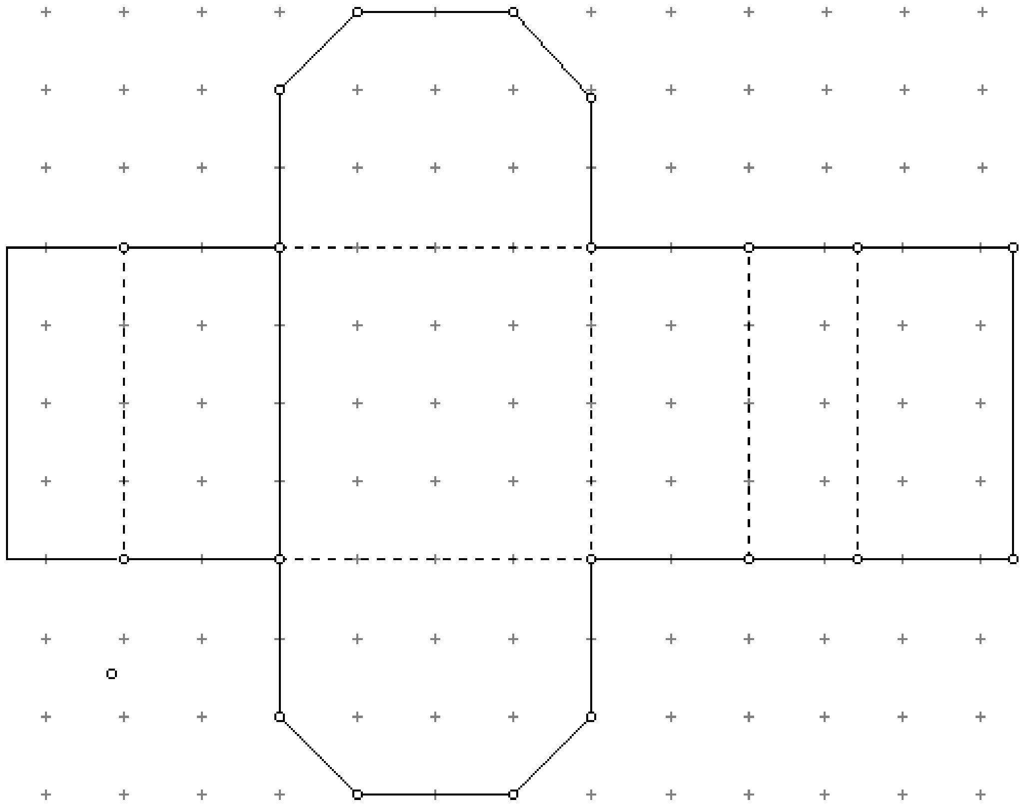

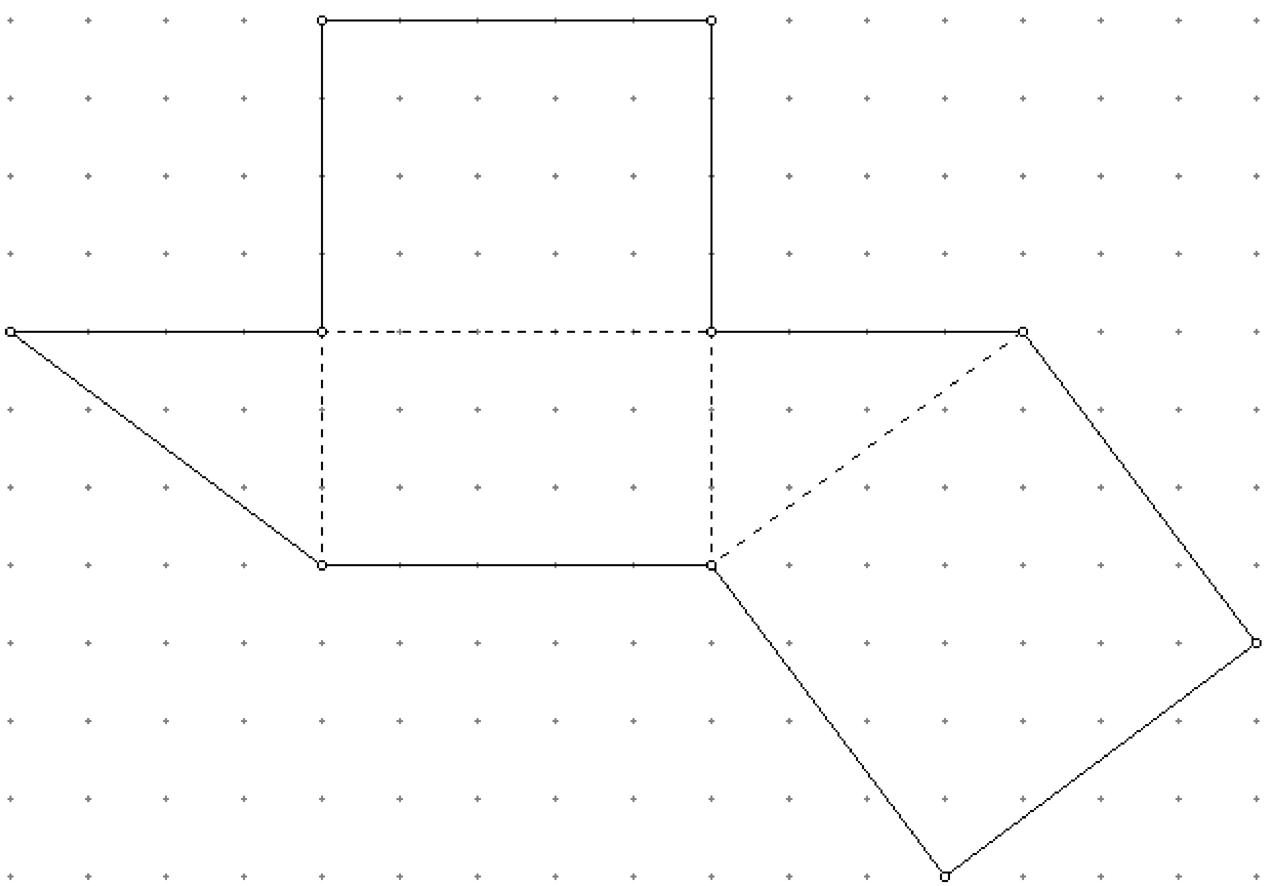

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Appendix G
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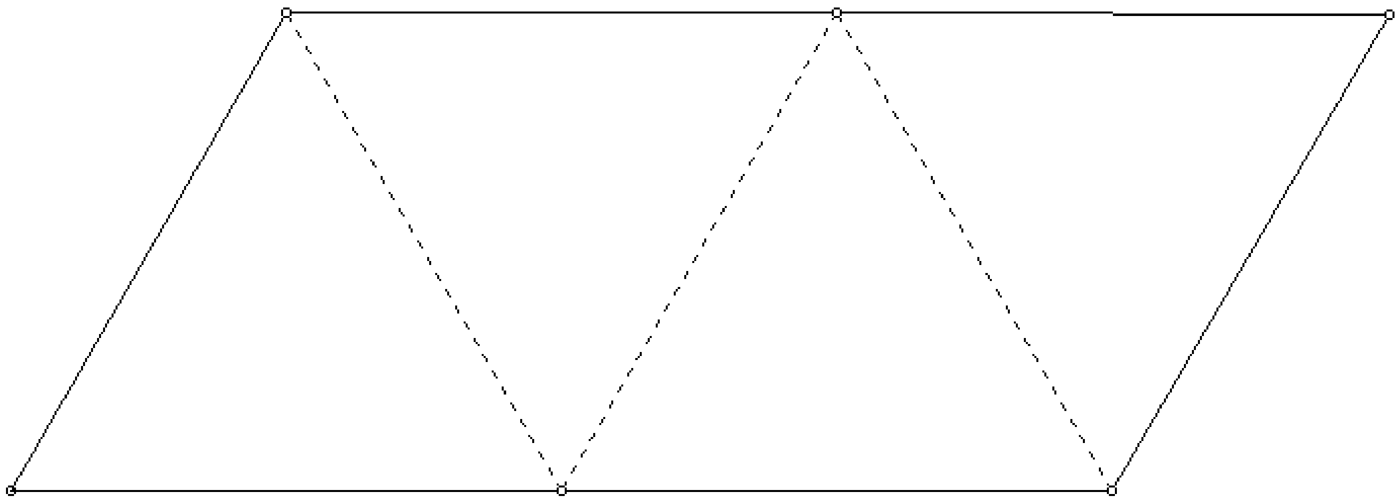

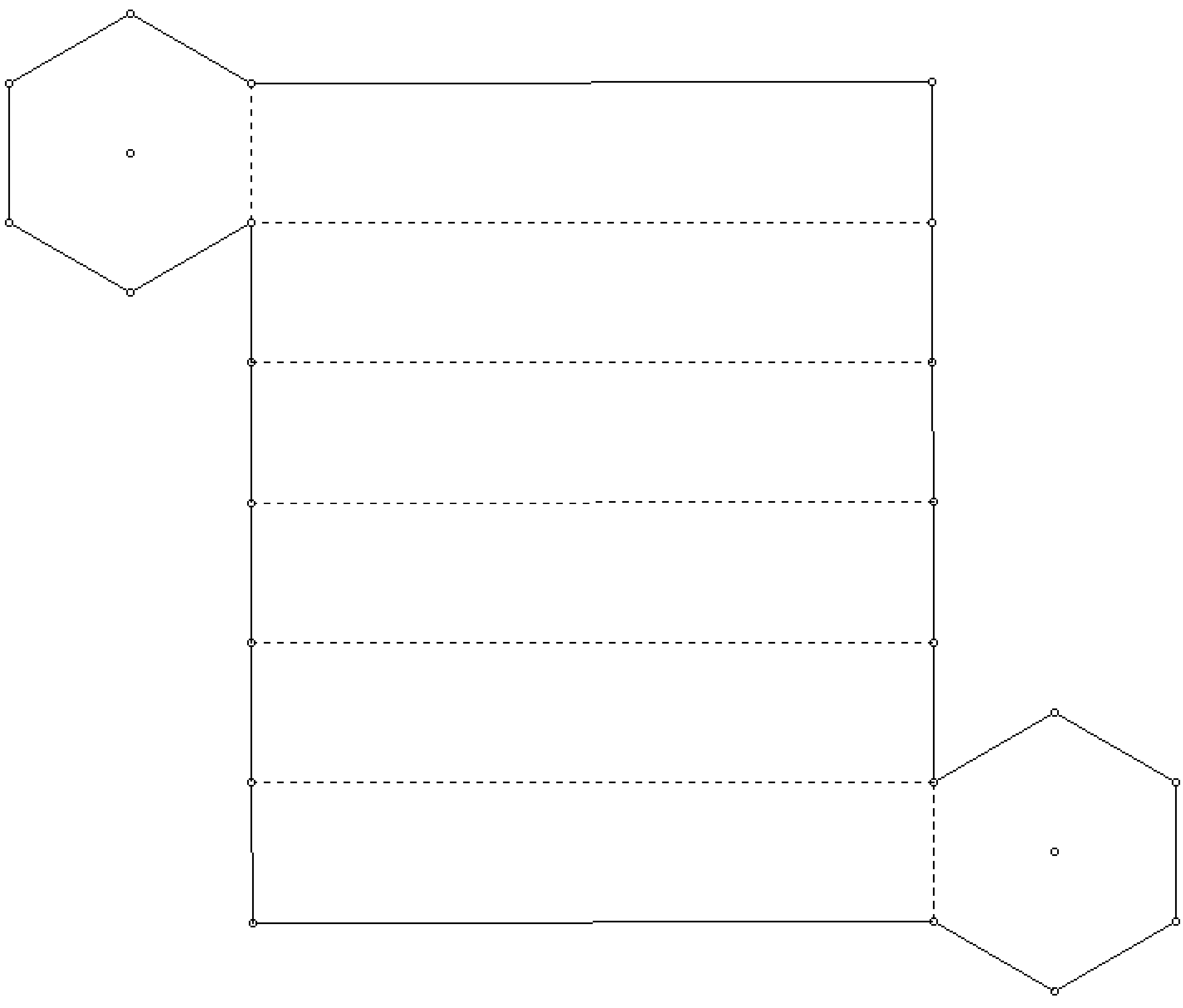

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Annotated References:

Internet Resources and Text Resources that are useful for exploring this geometry unit.

- Korbus, Linda – Perspective PowerPoint – 1999: Hinsdale South High School, Hinsdale, IL. *A Powerpoint® presentation focusing on Perspective drawing. The teacher presents famous art images using perspective and she demonstrates how to make a drawing in two-point perspective. This presentation is available at the Java Perspective site listed below.*

- Serra, Michael. (1997). Discovering geometry: An inductive approach (Second ed.). Berkeley, CA: Key Curriculum Press. *This textbook has many wonderful activities for exploring geometry.*

- **http://nctm.org** - National Council of Teachers of Mathematics. NCTM. See Principles and standards for school mathematics, as well as the Navigations and the on-line Illuminations.

- **http://www.mste.uiuc.edu/m2t2/geometry/perspective** – **Java Perspective**
  All the electronic files associated with drawings in perspective. They are interactive, web-based, and have downloadable documents and sketches.

- http://www.mste.uiuc.edu/dildine/sketches – **James P. Dildine's Java Geometry**
  Additional Java sketches that any student or teacher can access from the Web

- http://www.webmath.com – **Webmath**
  Web-based interactive mathematics resources (Dictionary, Formulas, etc…).

- http://www.math.com – **Math.com**
  Web-based interactive mathematics resources (Dictionary, Formulas, etc…)

- http://www.keypress.com – **Key Curriculum Press**
  Textbook and Software Publisher specializing in The Geometer's Sketchpad® and references for GSP.

Email questions and comments to
m2t2@mail.mste.uiuc.edu

M2T2

Geometry